27/09

Ultime proprietà

Sia $f: S \to T$ un'applicazione.

(1) $\forall X \subseteq S \implies X \subseteq f^{-1}(f(X))$

(2) $\forall Y \subseteq S \implies Y \supseteq f(f^{-1}(Y))$

Dim.

(1) $X \subseteq f^{-1}(f(X))$

$$\underbrace{x \in X}_{\Downarrow} \overset{?}{\implies} x \in f^{-1}(f(X))$$

$$f(x) \in f(X) \implies x \in \underbrace{f^{-1}(f(X))}$$

$$\Rightarrow f^{-1}(f(X)) = \{x \in S : f(x) \in f(X)\}$$

$$f^{-1}(*) = \{x \in S : f(x) \in *\}$$

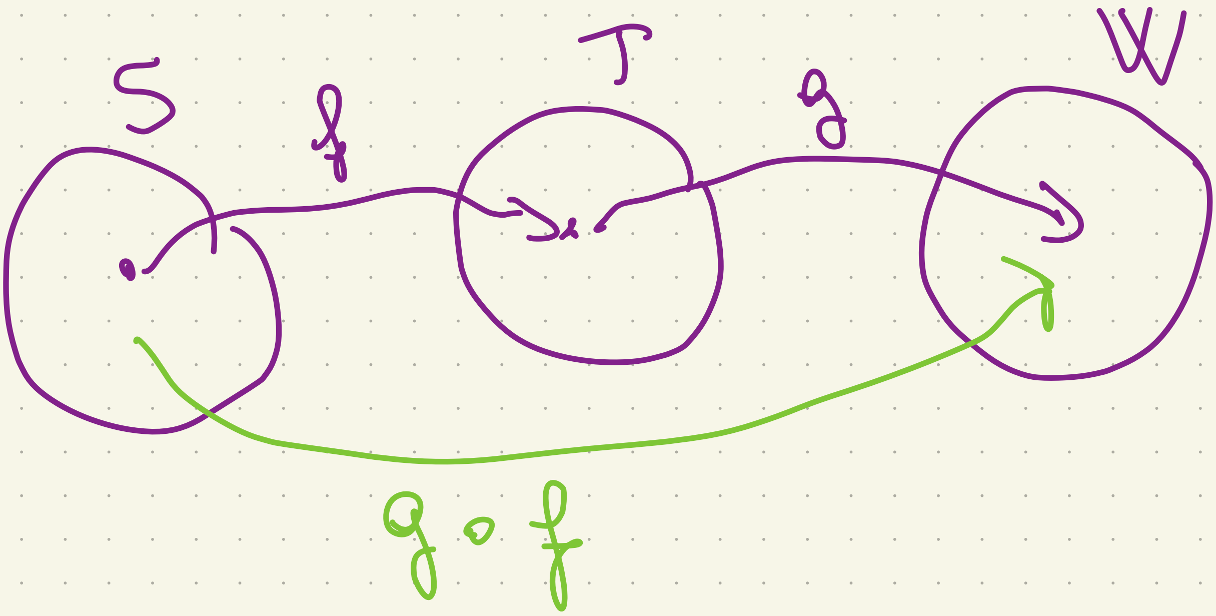
(2) per esercizio

Composizione tra funzioni

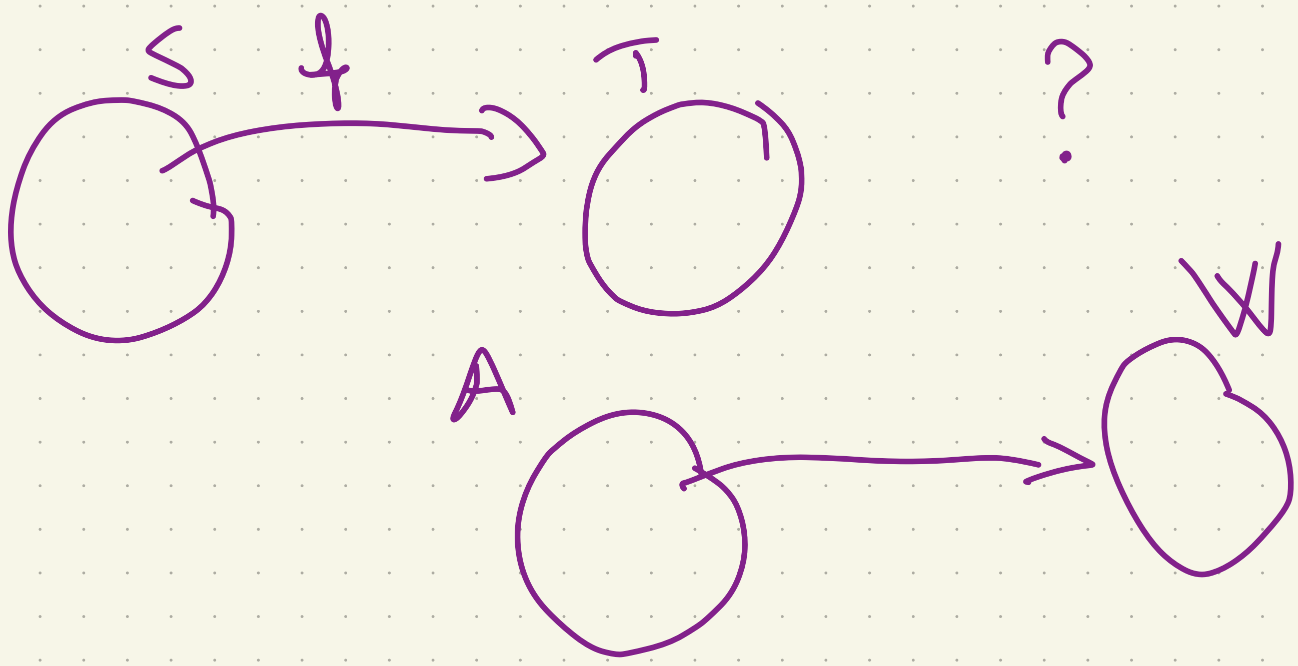
$f: S \to T$

$g: T \to W$

S, T, W insiemi



Se $g: A \to W$



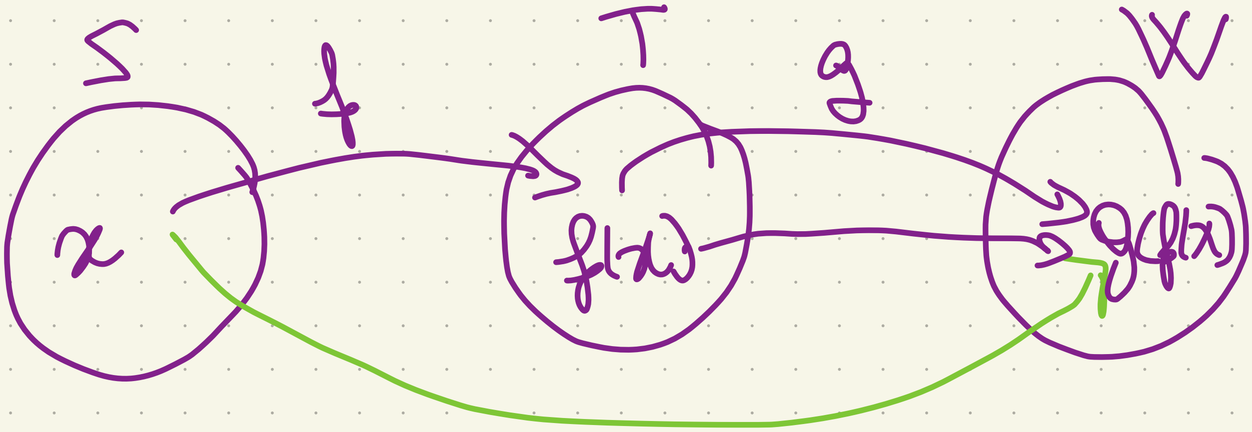
$g: T \to \boxed{W}$

$f: \boxed{S} \to T$

N.B. In generale $g \circ f \neq f \circ g$.

Quando può succedere che
$$g \circ f = f \circ g \,?$$

$$\begin{cases} f: A \to A \\ g: A \to A \end{cases}$$ I cond.



$$\begin{cases} g \circ f : S \longrightarrow W \\ \quad x \longmapsto g(f(x)) \end{cases}$$

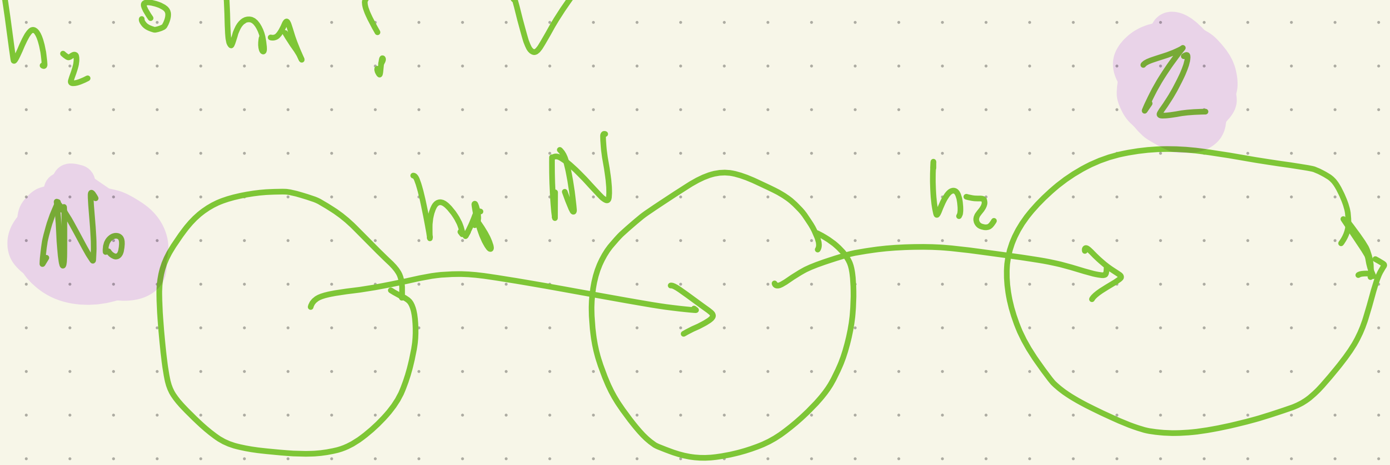
FUNZ COMPOSTA

ES.

- $h_1 : x \in \mathbb{N}_0 \longmapsto x^2 + 3 \in \mathbb{N}$
- $h_2 : x \in \mathbb{N} \longmapsto -3x + 5 \in \mathbb{Z}$

$h_1 \circ h_2$? ✗

$h_2 \circ h_1$? ✓



$$\underbrace{h_2 \circ h_1} : \mathbb{N}_0 \to \mathbb{Z}$$
$$x \mapsto h_2(h_1(x))$$

$$h_2(\underbrace{h_1(x)}) = h_2(x^2+3)$$
$$= -3(x^2+3)+5$$

$$h_2 : x \in \mathbb{N} \mapsto -3x+5 \in \mathbb{Z}$$
$$\square \mapsto -3\square+5$$
$$\text{marielaura} \mapsto -3(\text{marielaura})+5$$
$$x^2+3 \mapsto -3(x^2+3)+5$$

$$h_2 \circ h_1 : \mathbb{N}_0 \to \mathbb{Z}$$
$$x \mapsto -3(x^2+3)+5$$

Proprietà associativa

$f: S \to T$, $g: T \to W$

$h : W \to A$

$$h \circ (g \circ f) \overset{?}{=} (h \circ g) \circ f$$

Dim.

$$\boxed{h \circ (g \circ f)} \leftarrow (1)$$

$$\boxed{(h \circ g) \circ f} \leftarrow (2)$$

$(1) = (2)$

$$(1) \quad \underbrace{h \circ \underbrace{(g \circ f)}_{S \to W}}_{S \to A} : S \to A$$

$$(h \circ (g \circ f))(x) =$$

$$h((g \circ f)(x)) = h(g(f(x)))$$

$$(2)\quad \underbrace{\underbrace{(h \circ g)}_{T \to A} \circ f}_{S \to A} : S \to A$$

$$\big((h \circ g) \circ f\big)(x) =$$

$$(h \circ g)(f(x)) = h(g(f(x)))$$

$$\forall x \in S$$

<u>PROPRIETÀ</u> $f: S \to T$, $g: T \to V$

$\underline{f, g}$ iniettive $\Longleftrightarrow$ $g \circ f$ $\underline{\text{iniettiva}}$

(1) $\Longrightarrow$?

(2) $\Longleftarrow$?

PROPRIETÀ

Sia $f: S \to T$, $g: T \to V$

(1) f, g iniettive $\Rightarrow g \circ f$ iniettiva

(2) f, g suriettive $\Rightarrow g \circ f$ suriettiva

(3) f, g biettive $\Rightarrow g \circ f$ biettiva

Dim.

(1) Ipotesi f, g iniettive

Th: $g \circ f$ iniettiva

$g \circ f: S \to V$

$x_1, x_2 \in S$

$$\boxed{(g \circ f)(x_1) = (g \circ f)(x_2)}$$

Voglio dim. che $x_1 = x_2$

$(g \circ f)(x_1) = g(f(x_1))$

$(g \circ f)(x_2) = g(f(x_2))$

$g(f(x_1)) = g(f(x_2))$

g iniettiva $\left(g(y_1) = g(y_2) \Rightarrow y_1 = y_2 \right)$

$\Rightarrow f(x_1) = f(x_2)$

f iniettiva $\Rightarrow \boxed{x_1 = x_2}$

(2) f, g suriettive $\Longrightarrow$ $g \circ f$ suriettiva

$g \circ f : S \to V$

$f : S \to T$

$g : T \to V$

$\forall z \in V \; \exists x \in S : (g \circ f)(x) = z$

$(g(f(x)) = z)$

Per ipotesi, g e' sur.

$\forall z \in V \; \exists y \in T : g(y) = z$

f suriettiva

$\underline{\forall a \in T} \; \exists x \in S : f(x) = a$

$f: S \to T$ fun. $\iff f(S) = T$

se $a = y$

$f(x) = y$

$g(y) = z$

$\downarrow$

$g(f(x)) = z$

(2BIS)

TH:

$g \circ f$ suriettiva

$(g \circ f)(\underline{S}) = \underline{V}$

$g(f(S)) = g(T)$
$= V$

(3) f, g biettiva $\Rightarrow g \circ f$ biettiva

gratis (1) + (2)

PROPRIETÀ

(1) $g \circ f$ iniettiva $\Longrightarrow$ f iniettiva
~~g iniettiva~~

(2) $g \circ f$ suriettiva $\Longrightarrow$ g suriettiva

(3) $g \circ f$ biettiva $\Longrightarrow$ f iniettiva
g suriettiva

Dim. Per esercizio.

INVERSA

$f: S \longrightarrow T$ biettiva.

Ha senso definire l'applicazione

$f^{-1}: T \longrightarrow S$ tale che

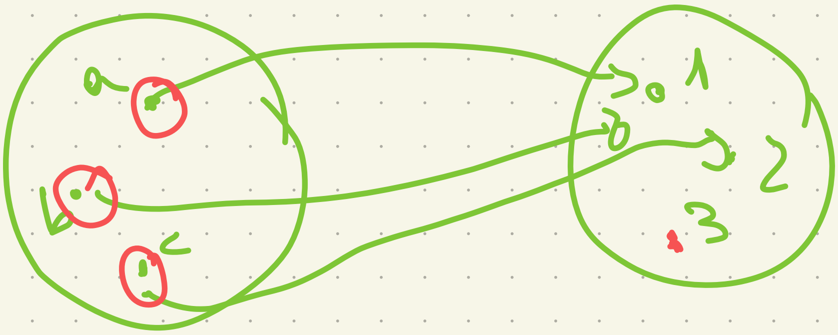
• $\underbrace{f \circ f^{-1}}_{\hookrightarrow \, id_S} : S \longrightarrow S$

• $f^{-1} \circ f : T \to T$

↳ id_T

$S = \{a, b, c\}$ $\quad T = \{1, 2, 3\}$

ES. $f = \{(a, 1), (b, 1), (c, 2)\}$



f non è iniettiva $\quad f(a) = 1,\ f(b) = 1$ ma $a \neq b$

f non è sur. $\quad f^{-1}(\{3\}) = \emptyset$

f^{-1} ha come dom $T = \{1, 2, 3\}$

$f^{-1} = \{(1, a), (1, b), (2, c)\}$

iniettività

$3 \to ?$

ESERCIZI

(1) $f: x \in \mathbb{Z} \longmapsto x-5 \in \mathbb{Z}$

Trovare l'inversa.

f iniettiva? ✓

$\forall x_1, x_2 \in \mathbb{Z}$

(I) $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$

(II) $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$

(I) $x_1 \neq x_2$

Se sommo -5 a entrambi i membri

ottengo $\underbrace{x_1 - 5}_{f(x_1)} \neq \underbrace{x_2 - 5}_{f(x_2)}$

$\Longrightarrow f(x_1) \neq f(x_2)$

f è suriettiva? ✓ $\left(\begin{array}{l} f: \mathbb{Z} \to \mathbb{Z} \\ x \mapsto x-5 \end{array} \right)$

$\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z} : f(x) = y$

$\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z} : x - 5 = y$

?

$x = y+5 \in \mathbb{Z}$

$f(y+5) = y+5-5 = y$

$\shortparallel$

$f(x)$

$\Rightarrow f$ biettiva

$f^{-1} : \mathbb{Z} \to \mathbb{Z}$

$\boxed{y} \mapsto y+5$

$f(x) = y \iff x - 5 = y$

$x = \boxed{y+5}$

$f \circ f^{-1} = id_{\mathbb{Z}}$

- $(f^{-1} \circ f = id_{\mathbb{Z}})$

$\underline{f: x \in \mathbb{Z} \longmapsto x-5 \in \mathbb{Z}}$

$f^{-1}: y \in \mathbb{Z} \longmapsto y+5 \in \mathbb{Z}$

$\underline{(f \circ f^{-1})}(x) =$

$= f(f^{-1}(x)) =$

$= f(x+5) = x+5-5 = \underline{x}$

(2) $h: \mathbb{Z} \longrightarrow \mathbb{N}_0$

$x \longmapsto |x|+4$

Iniettiva? Suriettiva?

IN

$h(-1) = |-1| + 4 = 1+4 = 5$

$h(1) = |1| + 4 = 1+4 = 5$

$h(-1) = h(1)$ ma $1 \neq -1$

SUR

$\forall y \in \mathbb{N}_0 \ \overset{?}{\exists} x \in \mathbb{Z} : h(x) = y$

$|x| + 4 = y$

$|x| = \boxed{y-4}$ $\quad x = y-4$

$y = 0$

$|x| = 0 - 4 = -4$

$|x| = y - 4$

≥ 0

$\Rightarrow y - 4 \geq 0 \Rightarrow y \geq 4$

(3) $g: x \in \mathbb{N} \longmapsto 2x-1 \in \mathbb{N}_0$

Esercizio. (Si può trovare g^{-1}?)