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INFORMAZIONI
SUL CORSO,
ESERCIZI,
DISPENSE

26/09

APPLICAZIONI

Siano S e T insiemi.

Un'applicazione da S in T è una relazione che ad ogni elemento di S associa uno e un solo elemento di T.

$$f: S \longrightarrow T$$

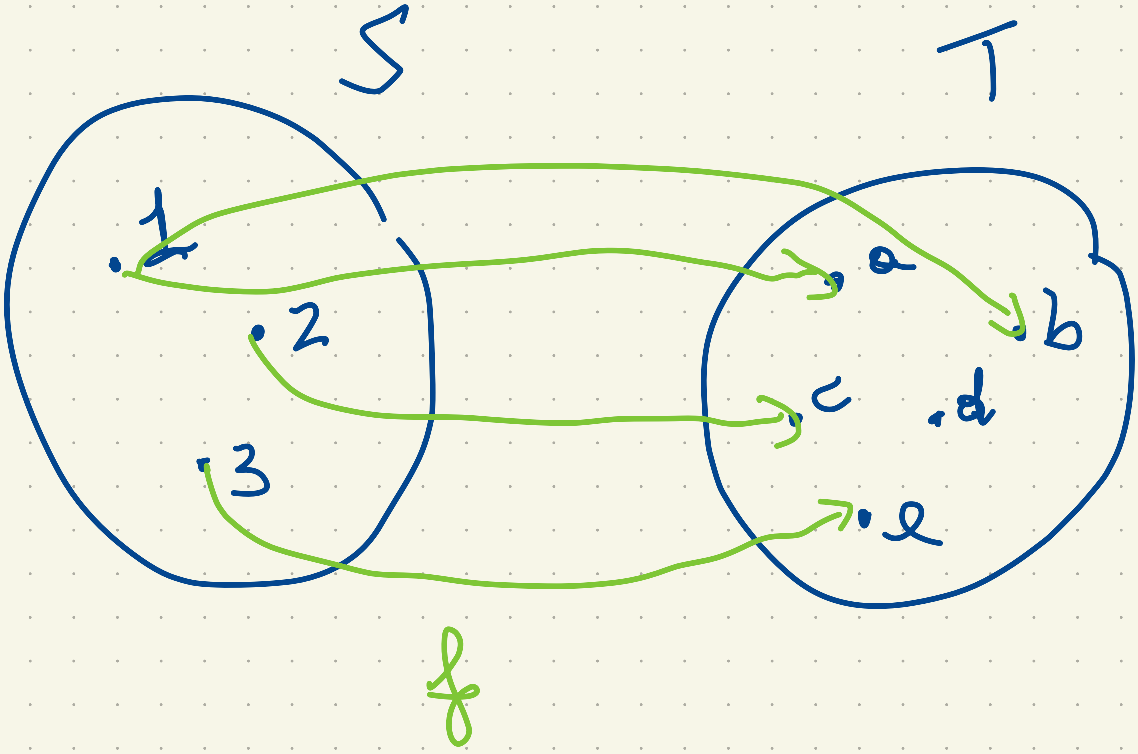
f è un'applicazione $\overset{def}{\Longleftrightarrow}$ $\forall x \in S \ \underbrace{\exists!}_{\text{esiste ed è unico}} y \in T:$

$x \, \underline{R} \, y \Longleftrightarrow \ldots$

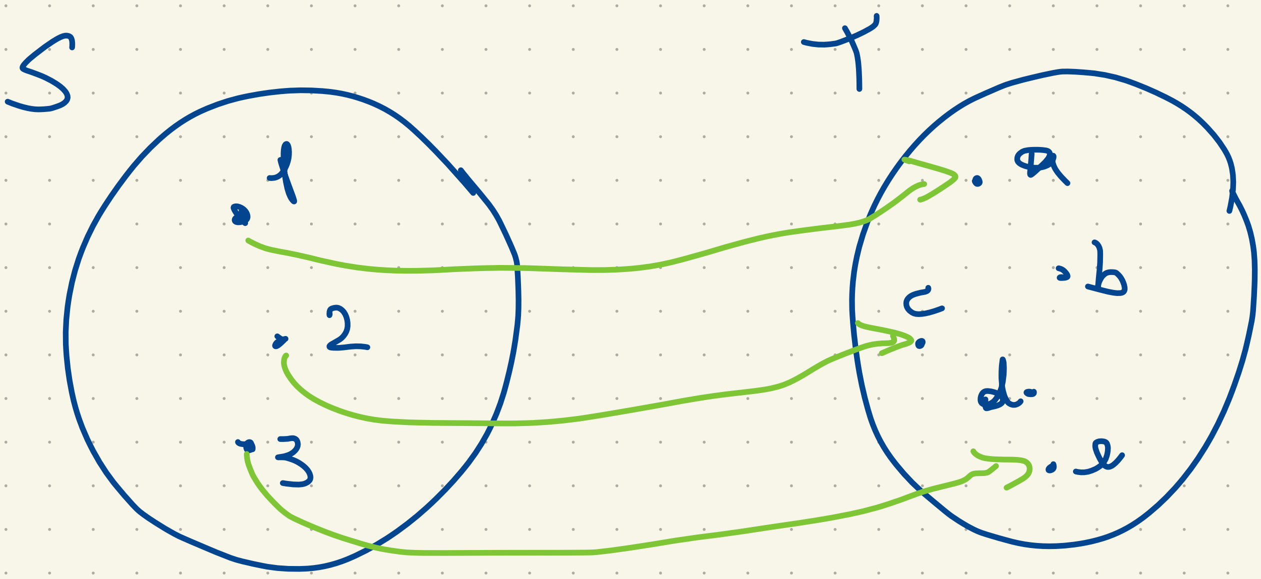
ES. $S = \{1, 2, 3\}$

$T = \{a, b, c, d, e\}$

$f: S \longrightarrow T$



f è un'applicazione? No
perché $f(1) = a$ e $f(1) = b$ ✗



f è un'applicazione? Sì

$f : \mathbb{Z} \to \mathbb{Z}$

$m \longmapsto \pm m$

f è un'applicazione? NO

perché $f(2) = 2$

$f(2) = -2$

<u>Oss</u>. S_1, S_2, T_1, T_2 insiemi

$f : S_1 \to T_1$ applicazione

$g : S_2 \to T_2$ applicazione

$$f = g \,? \quad \begin{cases} (1)\ S_1 = S_2 \\ (2)\ T_1 = T_2 \\ (3)\ f(x) = g(x) \quad \forall x \in S_1 \end{cases}$$

<u>ES</u>. S insieme, l'applicazione identica

$id_S : S \to S$

$s \longmapsto s$

Per definire un'applicazione servono 3 "condizioni":

- Dominio
- Codominio
- Legge

ES. $f: \mathbb{R} \to \mathbb{R}$

$x \longmapsto x^2$

$-1 \longmapsto 1$

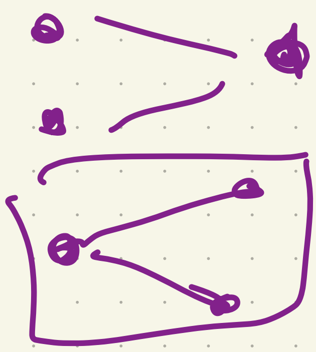
1

$f(1) = 1$

$f(-1) = 1$ ✓

$f(1) = 1$

$f(1) = 2$ ✗



no

ES. $\cdot\, S = \{\Delta, *, @\}$

$\cdot\, T = \{1, 2, 3\}$

$\cdot\, f(\Delta) = 1$ $\qquad f(\Delta) = 3$

$f(*) = 2$ $\qquad f(@) = 2$

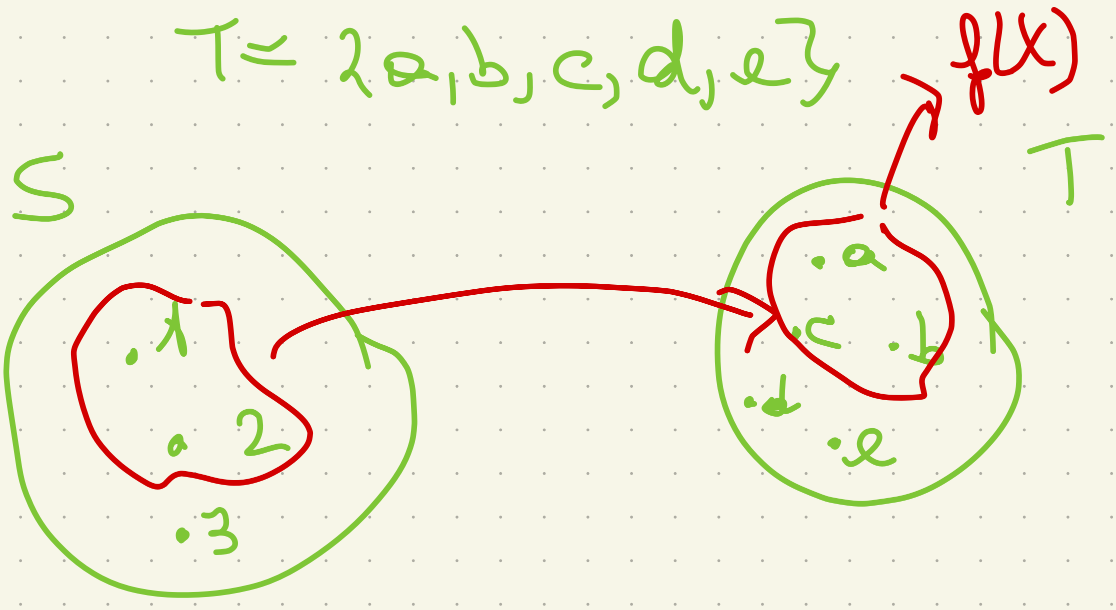
DEF. $f : S \to T$

- $X \subseteq S$

$$f(X) = \{ f(x) : x \in X \} \subseteq T$$

ES. $S = \{1,2,3\}$

$T = \{a,b,c,d,e\}$



$X = \{1,2\} \subseteq S$

- $X = S$

$$f(S) = \{ f(x) : x \in S \}$$

$\mathrm{Im} f$

ES.

$$f: \mathbb{Z} \to \mathbb{N}_0$$
$$x \mapsto x^2$$

$f(a) = a^2$

$f(1) = 1^2 = 1$

$X = \{1, 2, 3, -3\} \subseteq \mathbb{Z}$

$f(X) = f(\{1, 2, 3, -3\}) =$

$= \{f(1), f(2), f(3), f(-3)\}$

$= \{1, 4, 9, 9\}$

$= \{1, 4, 9\} \subseteq \mathbb{N}_0$

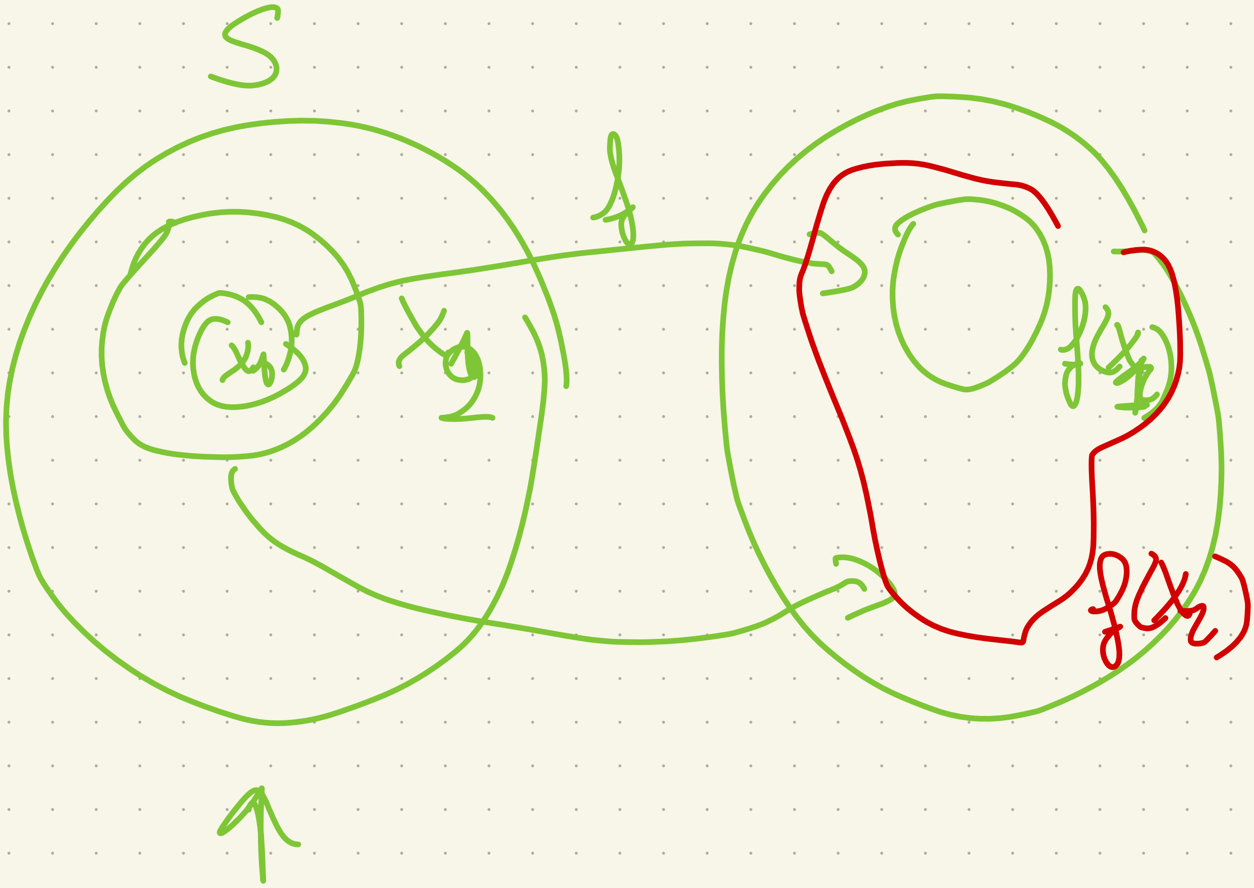
$X = \mathbb{Z}$

$f(\mathbb{Z}) = \{f(x) : x \in \mathbb{Z}\}$

$= \{x^2 : x \in \mathbb{Z}\}$

PROPRIETÀ Sia $f: S \to T$ applicazione
$X_1, X_2 \subseteq S$.

① $X_1 \subseteq X_2 \Rightarrow f(X_1) \subseteq f(X_2)$? SI



$X_1 \subseteq X_2$

② $f(X_1) \subseteq f(X_2) \Rightarrow X_1 \subseteq X_2$? NO

Dim.

(1) $X_1 \subseteq X_2 \Rightarrow f(X_1) \subseteq f(X_2)$

IPOTESI: $y \in f(X_1)$

TESI: $y \in f(X_2)$

$$y \in f(X_1) \iff \exists x \in X_1 : f(x) = y$$

$$f(X_1) = \{f(x) : x \in X_1\}$$

$\Rightarrow x \in X_2$

$$\Rightarrow \underbrace{f(x)}_{\in f(X_2)} = y \Rightarrow y \in f(X_2).$$

(2) FALSO

$f(X_1) \subseteq f(X_2) \Rightarrow X_1 \subseteq X_2$

Trovare f, S, T, X_1, X_2

$$f: \mathbb{Z} \to \mathbb{N}_0$$
$$x \mapsto x^2$$

Trovare X_1, X_2 tali che

$\underline{f(X_1) \subseteq f(X_2)}$ ma $X_1 \nsubseteq X_2$

X_1, $X_2 = \{-1, -2\}$

$\shortparallel$

$\{1, 2\}$

$f(X_1) = f(\{1, 2\}) = \{f(1), f(2)\}$
$= \{1, 4\}$

$f(X_2) = f(\{-1, -2\}) = \{f(-1), f(-2)\}$

$= \{1, 4\}$

$f(X_1) = f(X_2)$. In particolare

$f(X_1) \subseteq f(X_2)$.

<u>MA</u> $X_1 = \{1,2\} \neq \{-1,-2\}$

$$\begin{array}{c} \parallel \\ X_2 \end{array}$$

③ $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$

<u>Dim.</u> DOPPIA INCLUSIONE

(3.1) $f(X_1 \cup X_2) \subseteq f(X_1) \cup f(X_2)$

(3.2) $f(X_1 \cup X_2) \supseteq f(X_1) \cup f(X_2)$

(3.1), (3.2) Esercizio.

IPOTESI: $y \in f(X_1 \cup X_2)$

TESI: $y \in f(X_1) \cup f(X_2)$

$$\underset{\{f(x)\,:\,x \in X_1 \cup X_2\}}{\underset{\parallel}{y \in f(X_1 \cup X_2)}} \iff \exists\, x \in X_1 \cup X_2 :$$

$f(x) = y \implies x \in X_1$ oppure $x \in X_2$

$\implies f(x) \in f(X_1)$ oppure $f(x) \in f(X_2)$

$\implies \underbrace{f(x)}_{y} \in f(X_1) \cup f(X_2)$

$\implies y \in f(X_1) \cup f(X_2)$.

4) $f(X_1 \setminus X_2) \supseteq f(X_1) \setminus f(X_2)$

Esercizio

5) $f(X_1 \cap X_2) \;\; \begin{matrix} 1 \; = \; \times \\ 2 \; \subseteq \; \checkmark \\ 3 \; \supseteq \; \times \end{matrix} \;\; f(X_1) \cap f(X_2)$

Dim. 2 ✓

$y \in f(X_1 \cap X_2)$ vogliamo dim. che

$y \in f(X_1) \cap f(X_2)$.

$\Rightarrow \exists \underbrace{x \in X_1 \cap X_2}: f(x) = y$

$\Downarrow$

$\underbrace{x \in X_1}$ e $x \in X_2$

$\Downarrow \qquad \Downarrow$

$f(x) \in f(X_1)$ e $f(x) \in f(X_2)$

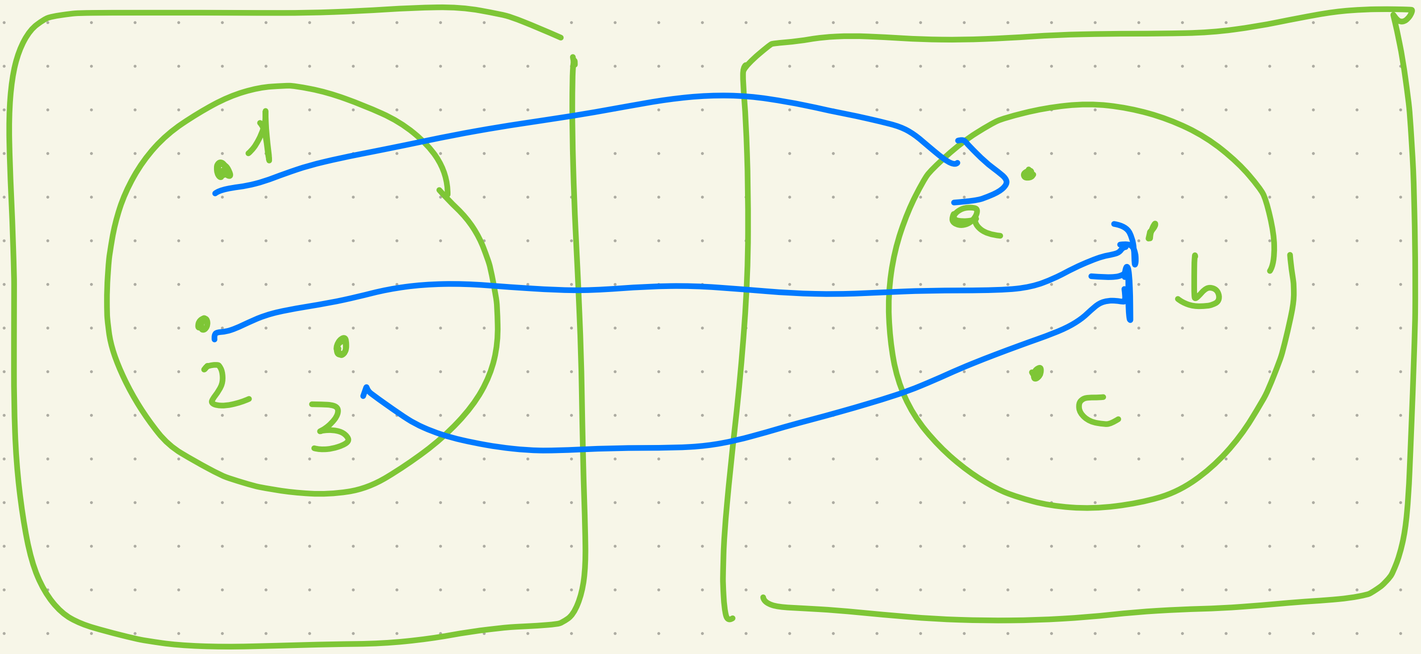
$\Rightarrow f(x) \in f(X_1) \cap f(X_2)$

$\| $

y

$\Rightarrow y \in f(X_1) \cap f(X_2)$.

$S = \{1, 2, 3\}$ $\quad$ $T = \{a, b, c\}$



$f(1) = a$

$f(2) = b$

$f(3) = b$

Trovare X_1 e X_2 tali che

$f(X_1) \cap f(X_2) \not\subseteq f(X_1 \cap X_2)$

$X_1 = \{1, 2\}$

$X_2 = \{2, 3\}$

$X_1 \cap X_2 = \{2\}$

$f(X_1) = \{f(1), f(2)\} = \{a, b\}$

$f(X_2) = \{f(1), f(3)\} = \{a, b\}$

$$f(X_1) \cap f(X_2) = \{a,b\} \cap \{a,b\}$$
$$= \{a,b\}$$

$$f(X_1 \cap X_2) = f(\{2\}) = \{a\}$$

$$f: S \longrightarrow T$$

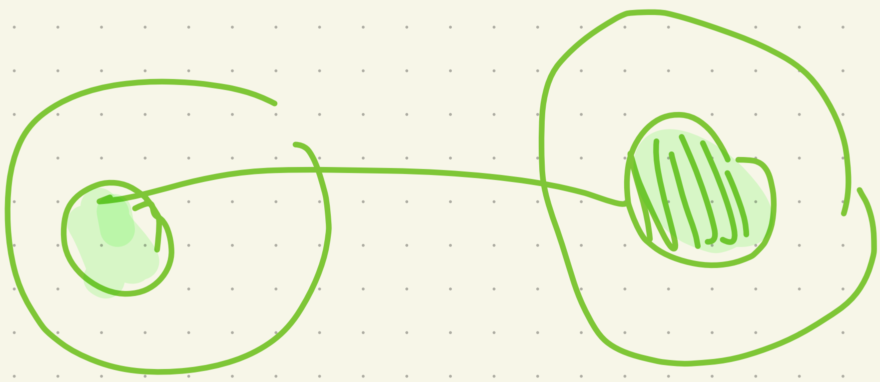
DEF. Controimmagine di Y tramite f

$Y \subseteq T$

tutti gli elementi di S la cui immagine appartiene ad Y.

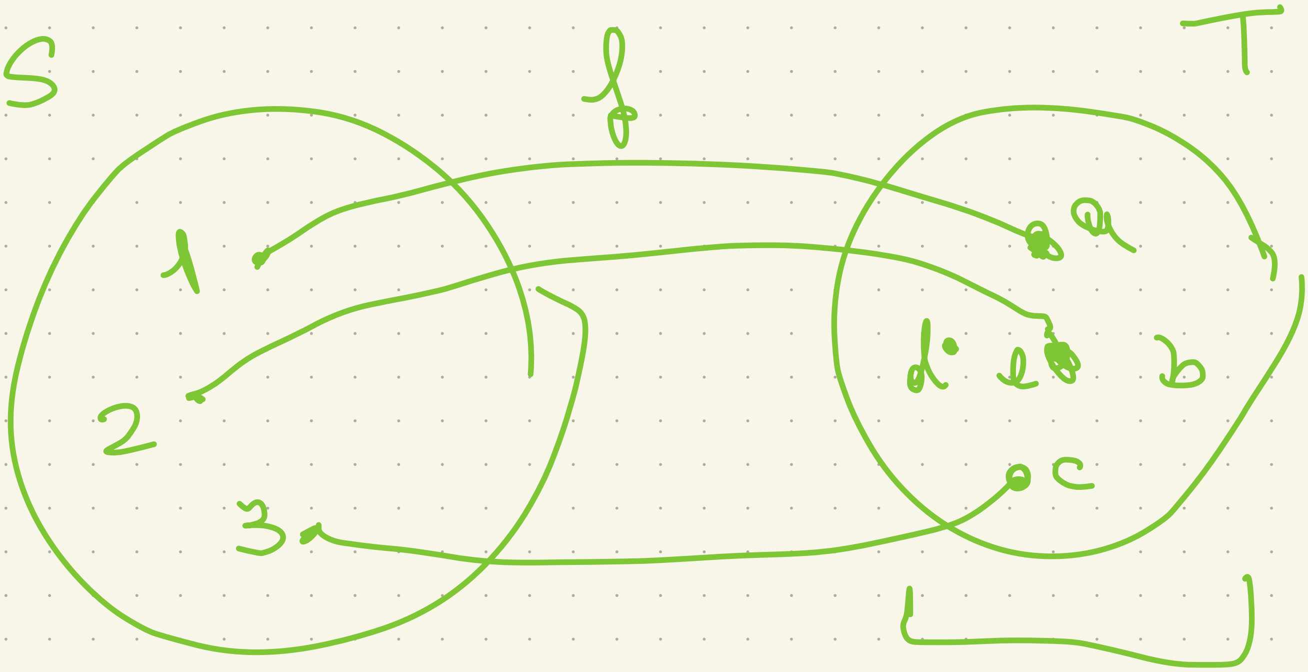
$$f^{-1}(Y) = \{x \in S : f(x) \in Y\}$$

$$f^{-1}(Y) \subseteq S$$



ES. $S = \{1, 2, 3\}$

$T = \{a, b, c, d, e\}$



$$Y = \{a, e\}$$

$$f^{-1}(Y) = \{1, 2\}$$

$$Y = \{d\}$$

$$f^{-1}(Y) = \emptyset$$

PROPRIETÀ

Sia $f: S \to T$ e siano $Y_1, Y_2 \subseteq T$.

① $Y_1 \subseteq Y_2 \Longrightarrow f^{-1}(Y_1) \subseteq f^{-1}(Y_2)$

② $Y_1 \subseteq Y_2 \not\Longleftarrow f^{-1}(Y_1) \subseteq f^{-1}(Y_2)$

<u>Dim.</u> ①

$$\underbrace{x \in f^{-1}(Y_1)}_{\text{IP.}} \Longrightarrow \underbrace{x \in f^{-1}(Y_2)}_{\text{TH.}}$$

$$f^{-1}(Y_1) = \{x \in S : f(x) \in Y_1\}$$

$f(x) \in Y_1$

ma $Y_1 \subseteq Y_2 \Longrightarrow f(x) \in Y_2$

$\Longrightarrow x \in f^{-1}(Y_2)$

② $f: \mathbb{Z} \to \mathbb{N}_0$ $\quad Y_1 = \{1, 3\}$

$x \longmapsto x^2$ $\quad Y_2 = \{1\}$

tali che $f^{-1}(y_1) \subseteq f^{-1}(y_2)$ ma $y_1 \not\subseteq y_2$.

$$f^{-1}(\{1,3\}) = \{f^{-1}(1)\} \cup \{f^{-1}(3)\}$$

$$f^{-1}(\{1\}) = \{x \in \mathbb{Z} : f(x) = 1\}$$

$$= \{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$$

$$f^{-1}(\{3\}) = \{x \in \mathbb{Z} : f(x) = 3\}$$

$$= \{\underline{x \in \mathbb{Z}} : x^2 = 3\} = \emptyset$$

$$f^{-1}(\{1,3\}) = \{1, -1\} \cup \{\emptyset\}$$

$$= \{1, -1\}$$

$$f^{-1}(y_2) = f^{-1}(\{1\}) = \{1, -1\}$$

$f^{-1}(y_1) \subseteq f^{-1}(y_2)$ ma

$y_1 \not\subseteq y_2$.

③ $f^{-1}(Y_1) \cap f^{-1}(Y_2) = f^{-1}(Y_1 \cap Y_2)$

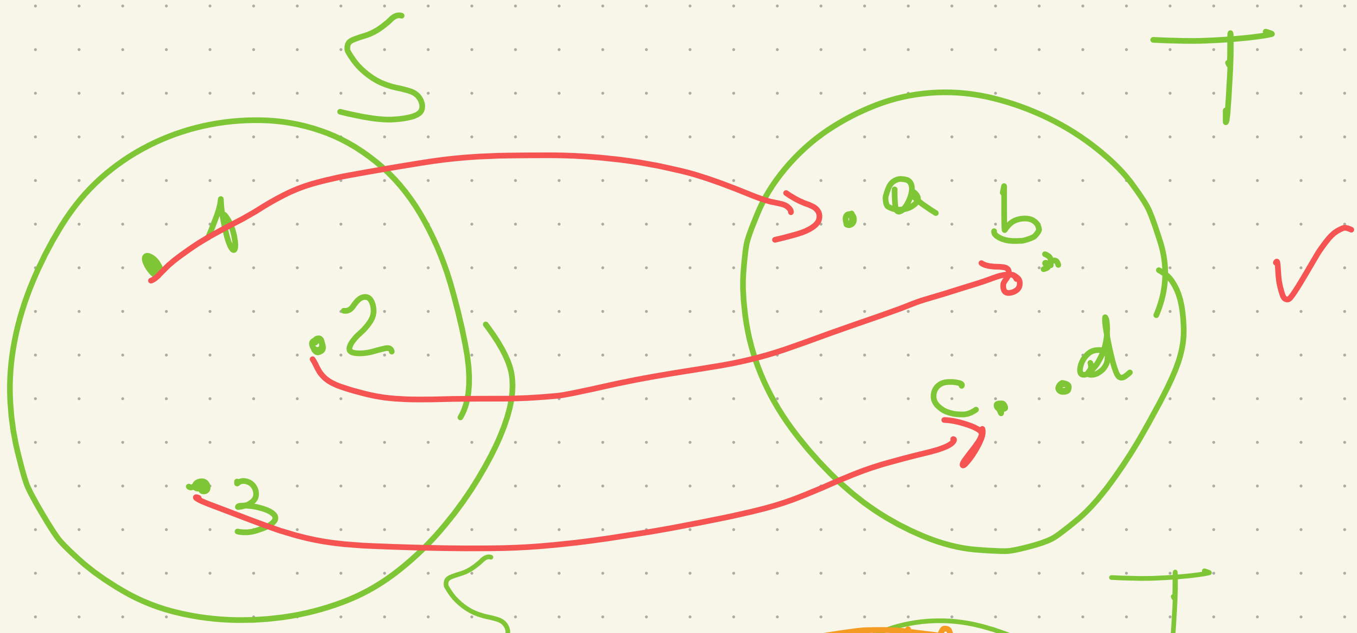
④ $f^{-1}(Y_1) \cup f^{-1}(Y_2) = f^{-1}(Y_1 \cup Y_2)$

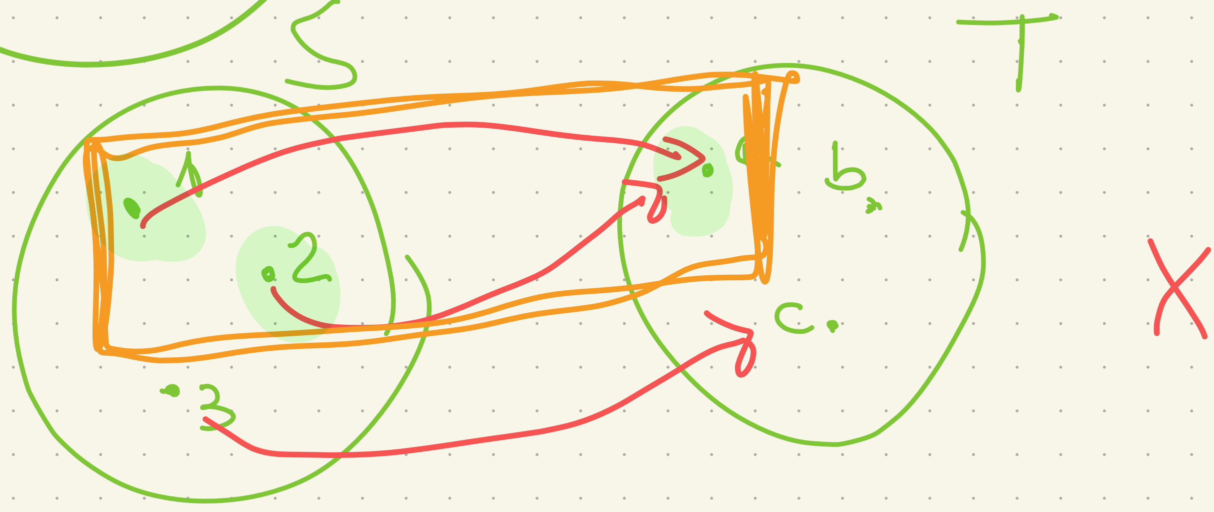
⑤ $f^{-1}(Y_1) \setminus f^{-1}(Y_2) = f^{-1}(Y_1 \setminus Y_2)$

Esercizio.

TIPI DI FUNZIONE $f: S \to T$

- Iniettiva





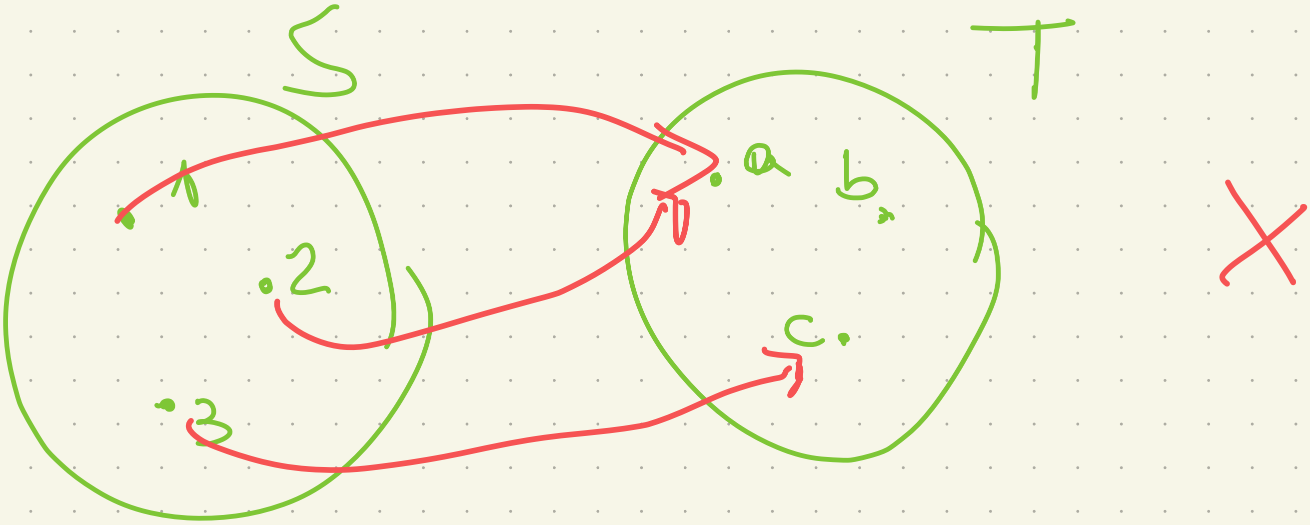
$x_1, x_2 \in S$

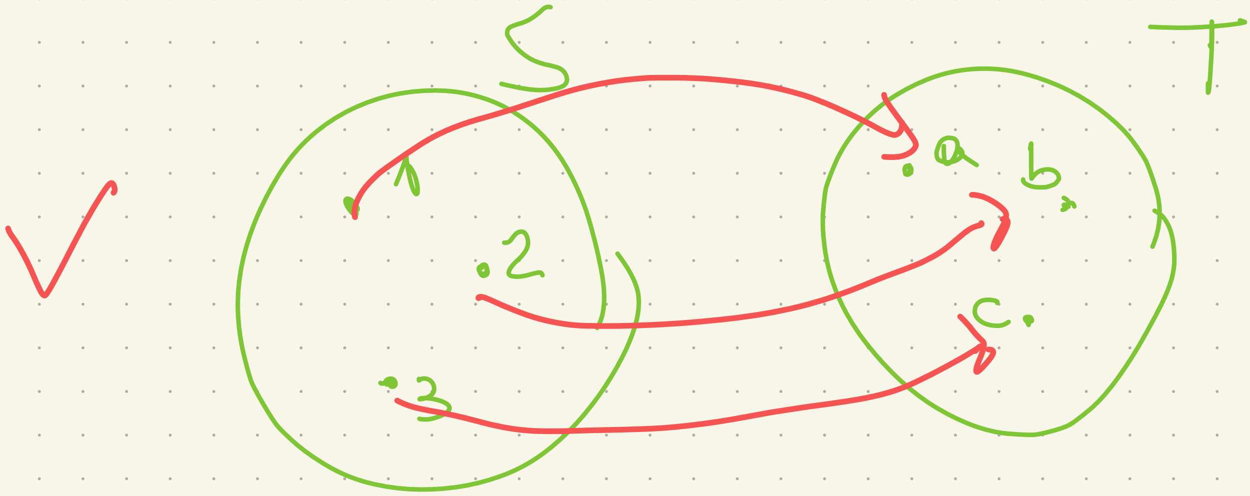
- $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

equivalente a

- $f(x_1) = f(x_2) \implies x_1 = x_2$

- <u>Suriettiva</u>





$\forall y \in T \ \exists x \in S : f(x) = y$

- <u>Biettiva</u> = SUR + INIET

$\forall y \in T \ \exists! x \in S : f(x) = y$

<u>ESERCIZIO</u>

- $f : \mathbb{N} \to \mathbb{Z}$
  $x \mapsto -x-5$

(1) $f(\{1,2,3\})$

(2) $f^{-1}(\{0,-1,-2,1\})$

(3) f è iniettiva?

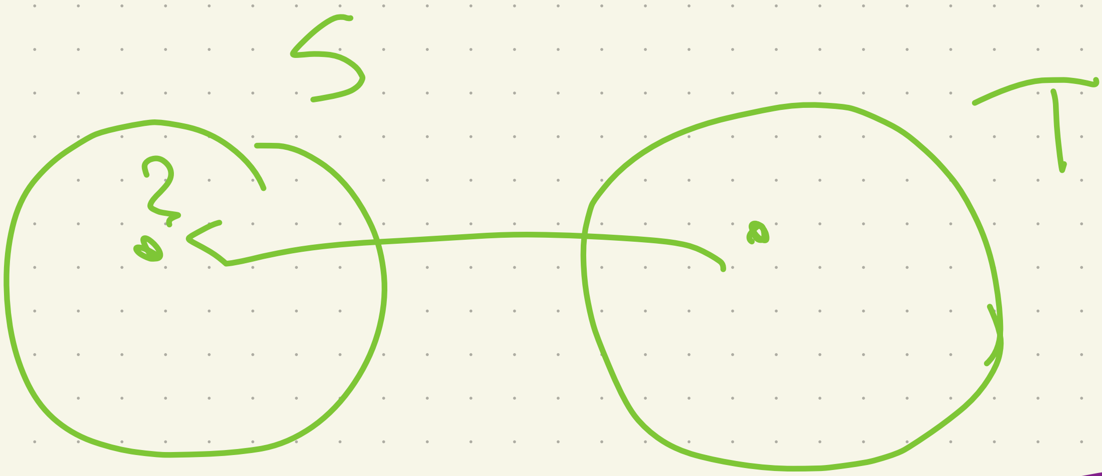
(1) $f(\{1,2,3\}) = \{f(1), f(2), f(3)\}$
$= \{-6, -7, -8\}$

(2) $f^{-1}(\{0,-1,-2,1\} =$

$= f^{-1}(\{0\}) \cup f^{-1}(\{-1\}) \cup f^{-1}(\{-2\}) \cup f^{-1}(\{1\})$

$f^{-1}(\{0\}) = \{x \in \mathbb{N} : f(x) = 0\}$

$= \{x \in \mathbb{N} : -x-5 = 0\} = \emptyset$



(3) È iniettiva?

$x_1, x_2 \in S$, f in $\overset{def}{\iff}$

$\boxed{f(x_1) = f(x_2)}$ IP

$\Downarrow$

$\boxed{x_1 = x_2}$ TH

$f: S \to T$
f è suriettiva
$\Updownarrow$
$f^{-1}(\{y\}) \neq \emptyset$
$\forall y \in T$

$f(x_1) = -x_1 - 5$

$f(x_2) = -x_2 - 5$

$-x_1 - 5 = -x_2 - 5$

$\Rightarrow x_1 = x_2$ ✓

• $g : \mathbb{Z} \longrightarrow \mathbb{N}$

$x \longmapsto 5x^2 + 4$

(1) g è in?

(2) g è sur?

(1) NO

$g(-1) = 9$

$g(1) = 9$

$g(-1) = g(1) \Rightarrow -1 = 1$

NO

(2) $\boxed{\forall y \in \mathbb{N}\ \exists x \in \mathbb{Z} : \underbrace{\overline{g(x)}}\ = y}$

$$5x^2 + 4 = y$$

$$x^2 = \frac{y-4}{5} \quad (y \geq 4)$$

$$x = \pm\sqrt{\frac{y-4}{5}} \in \mathbb{Z}?$$

$y = 1$

$$g^{-1}(\{1\}) = \{x \in \mathbb{Z} : g(x) = 1\}$$

$$= \{x \in \mathbb{Z} : 5x^2 + 4 = 1\}$$

$$= \{x \in \mathbb{Z} : \underbrace{5x^2} = \underbrace{-3}\} = \emptyset$$

- $h : \mathbb{N} \to \mathbb{N}$ (1) h è in.?

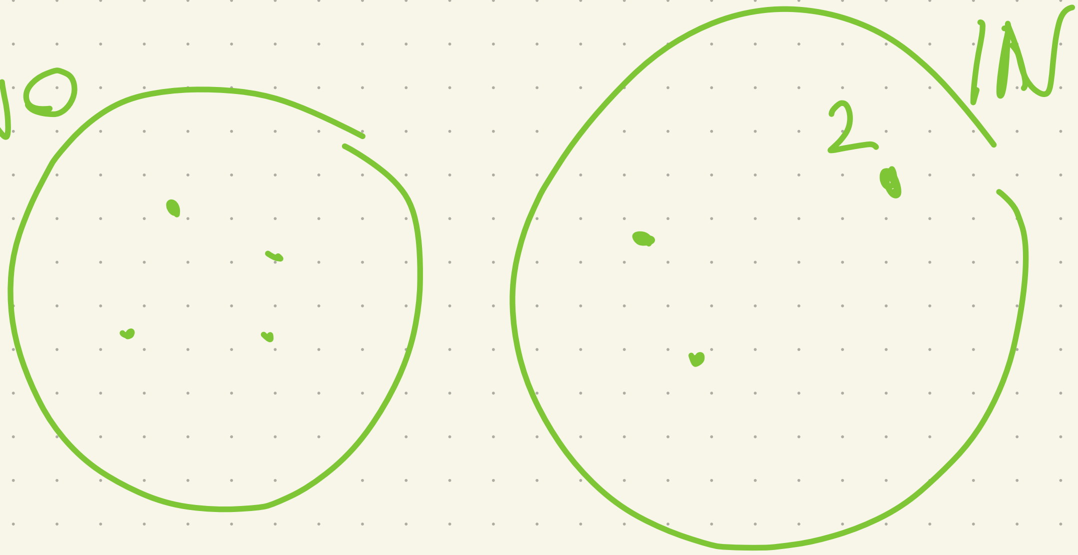
  $x \mapsto 1$ (2) h è sur?

(1) NO

$h(1) = 1$

$h(2) = 1$ ma $1 \neq 2$

(2) NO



$$h^{-1}(\{2\}) = \{x \in \mathbb{N} : h(x) = 2\}$$
$$= \{x \in \mathbb{N} : 1 = 2\}$$
$$= \emptyset$$

APP. COSTANTE

$h : S \to S$

$x \mapsto k$

$k \in S$ fissato.