Lezione 03-10

martedì 3 ottobre 2023 11:07

TEOREMA - $f: S \to T$, $A_1, A_2 \subseteq S$ - Si ha

1) Se $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$
2) $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$
3) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$
4) $f(A_1) \setminus f(A_2) \subseteq f(A_1 \setminus A_2)$

Dim

1) hp: $A_1 \subseteq A_2$

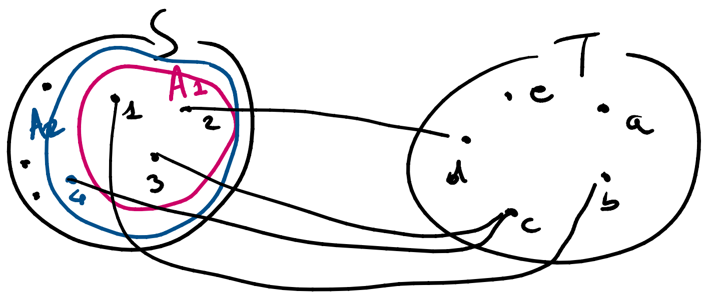
Th: $f(A_1) \subseteq f(A_2)$

$f(A_1) = \{ y \in T \mid \exists x \in A_1 \text{ e } f(x) = y \} = \{ f(x) \mid x \in A_1 \}$

Se $y \in f(A_1)$ allora $\exists x \in A_1$ t.c. $f(x) = y$

Poiché $A_1 \subseteq A_2$, si ha che $x \in A_1 \Rightarrow x \in A_2 \Rightarrow$

$f(x) \in \{ y \in T \mid \exists x \in A_2 \text{ e } f(x) = y \} = f(A_2)$



$f(A_1) = \{ f(1), f(2), f(3) \} = \{ b, c, d \}$

$f(A_2) = \{ f(1), f(2), f(3), f(4) \} = \{ b, c, d \}$

2) $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$

$x \in f(A_1) \cup f(A_2) \Leftrightarrow x \in f(A_1)$ oppure $x \in f(A_2)$

$\Leftrightarrow \exists a \in A_1$ t.c. $x = f(a)$ oppure $\exists b \in A_2$ t.c. $x = f(b)$

$\Leftrightarrow \exists c \in A_1 \cup A_2$ t.c. $x = f(c)$ -

3) $f(A_1\cap A_2)\subseteq f(A_1)\cap f(A_2)$

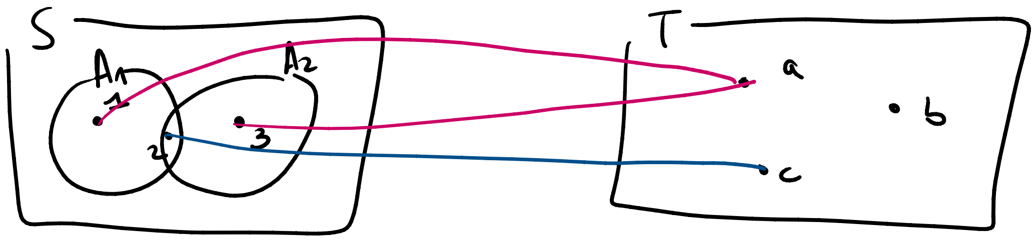
hp: $x\in f(A_1\cap A_2)$

th: $x\in f(A_1)\cap f(A_2)$

$x\in f(A_1\cap A_2) \iff \exists\, a\in A_1\cap A_2$ t.c. $f(a)=x$

$\iff \exists\, a$ t.c. $a\in A_1$ e $a\in A_2$ e $f(a)=x$

$\Rightarrow$ $x\in f(A_1)$ e $x\in f(A_2)$ $\Rightarrow x\in f(A_1)\cap f(A_2)$



$A_1=\{1,2\}$ $\quad f(A_1)=\{f(1),f(2)\}=\{a,c\}$

$A_2=\{2,3\}$ $\quad f(A_2)=\{f(2),f(3)\}=\{a,c\}$

$A_1\cap A_2=\{2\}$

$f(A_1)\cap f(A_2)=\{a,c\}$

$f(A_1\cap A_2)=\{f(2)\}=\{c\}$

4) $f(A_1)\setminus f(A_2)\subseteq f(A_1\setminus A_2)$

hp: $x\in f(A_1)\setminus f(A_2)$

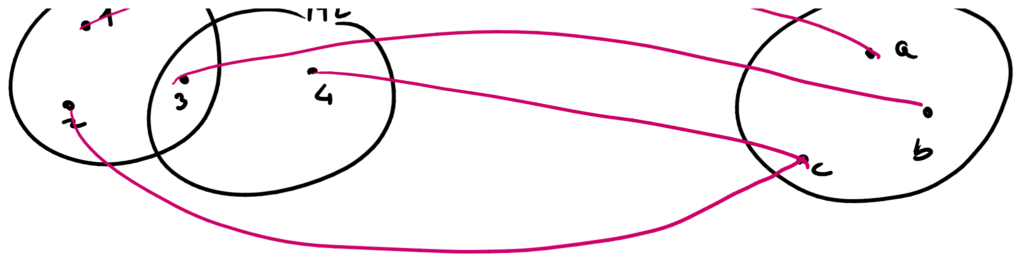
th: $x\in f(A_1\setminus A_2)$

$x\in f(A_1)\setminus f(A_2) \iff x\in f(A_1)$ e $x\notin f(A_2)$

$\iff \underline{\exists}\, a\in A_1$ tc $f(a)=\underline{x}$ e $\nexists\, b\in A_2$ tc $f(b)=\underline{x}$

$\Rightarrow \exists\, c\in A_1\setminus A_2$ tc $f(c)=x$.

$\Rightarrow x\in f(A_1\setminus A_2)$.



$A_1 = \{1, 2, 3\}$

$A_2 = \{3, 4\}$

$A_1 \setminus A_2 = \{1, 2\}$

$f(A_1 \setminus A_2) = \{f(1), f(2)\} = \{a, c\}$

$f(A_1) = \{f(1), f(2), f(3)\} = \{a, b, c\}$

$f(\underline{A_2}) = \{f(3), f(4)\} = \{b, c\}$

$f(A_1) \setminus f(A_2) = \{\underline{a}\}$

TEOREMA

Sia $f: S \to T$, $B_1, B_2 \subseteq T$

1) se $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

2) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

3) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

4) $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$

5) $\forall A \subseteq S$ si ha $A \subseteq f^{-1}(f(A))$

6) $\forall B \subseteq T$ si ha $f(f^{-1}(B)) \subseteq B$

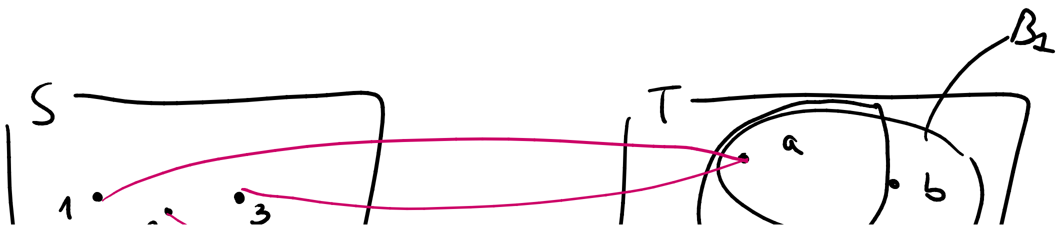
Dim

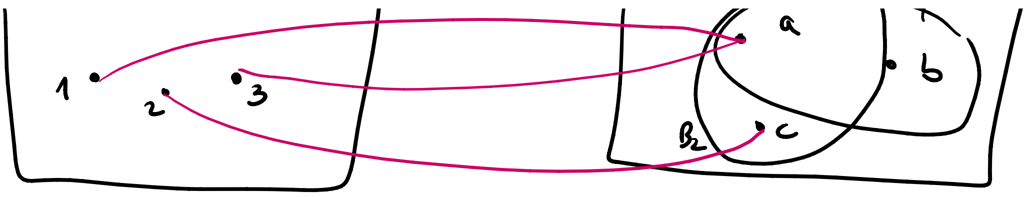
2) $f^{-1}(B_1) = \{a \in S \mid f(a) \in B_1\}$

$x \in f^{-1}(B_1 \cap B_2) \Leftrightarrow \exists a \in S$ t.c. $f(x) = a \in B_1 \cap B_2$

$\Leftrightarrow \exists a \in S$ tc $a = f(x) \in B_1$ e $a = f(x) \in B_2$

$\Leftrightarrow x \in f^{-1}(B_1) \cap f^{-1}(B_2)$





$B_1 = \{a, b\}$

$B_2 = \{a, c\}$

$f^{-1}(B_1) = \{1, 3\}$

$f^{-1}(B_2) = \{1, 2, 3\}$

$f^{-1}(B_1) \cap f^{-1}(B_2) = \{1, 3\}$

$B_1 \cap B_2 = \{a\}$

$f^{-1}(B_1 \cap B_2) = \{1, 3\}$

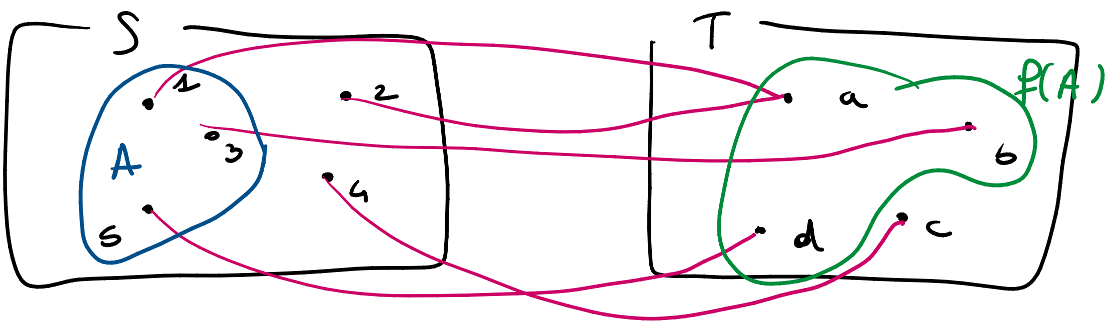
5) $\forall A \subseteq S \qquad A \subseteq f^{-1}(f(A))$

$x \in A \Rightarrow x \in f^{-1}(f(A))$

Se $x \in A$, allora $\underbrace{f(x) \in f(A)}_{*}$ quindi considero

$f^{-1}(f(A)) = \{a \in S \mid f(a) \in f(A)\}$

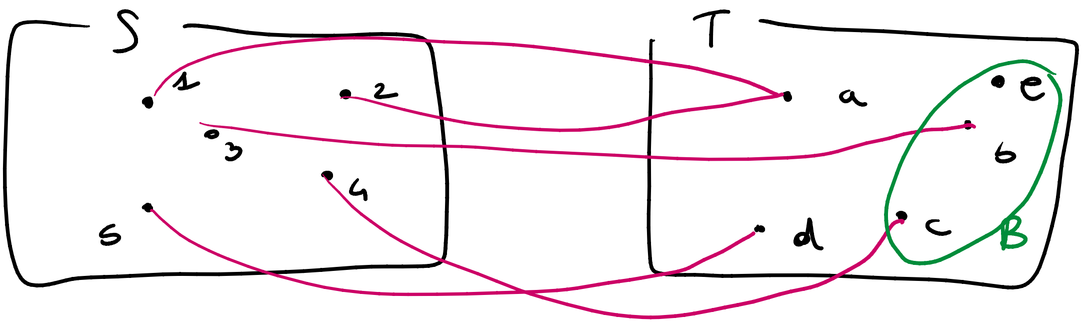
$\Rightarrow$ x soddisfa la condizione* $\Rightarrow x \in f^{-1}(f(A))$



$f(A) = \{f(1), f(5), f(3)\} = \{a, b, d\}$

$f^{-1}(f(A)) = f^{-1}(\{a, b, d\}) = \{1, 3, 5, 2\}$ $\rightarrow A \subsetneq f^{-1}(f(A))$

6) $\forall B \subseteq T \qquad f(f^{-1}(B)) \subseteq B$



$B = \{e, b, c\}$

$f^{-1}(B) = \{3, 4\}$

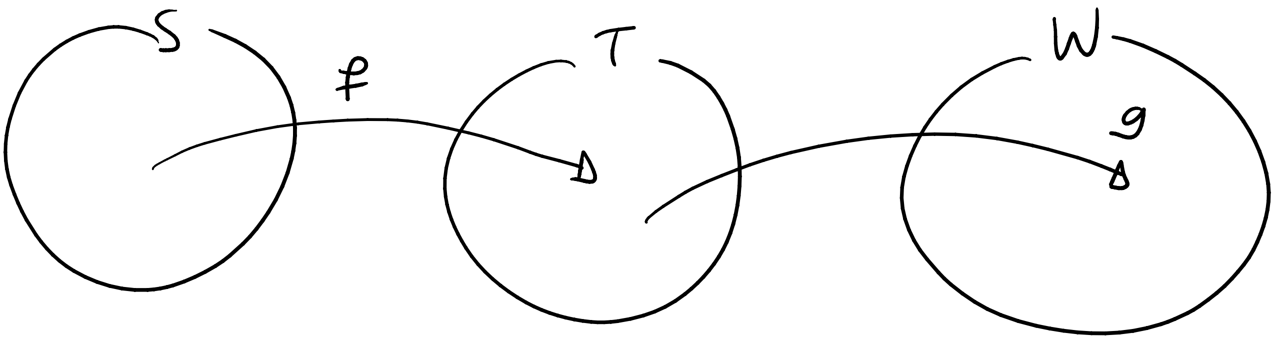
$f(f^{-1}(B)) = \{f(3), f(4)\} = \{b, c\}$

$f(f^{-1}(B)) \neq B$

<u>DEF</u> - $f: S \to T$ e $g: T \to W$

Chiamiamo COMPOSTA (o composizione) di f e g la funzione $g \circ f : S \to W$ definita da

$$(g \circ f)(s) = g(f(s))$$



<u>ES</u> - $S = \mathbb{N}$
$T = \mathbb{R}$
$W = \mathbb{R}$

$f : \mathbb{N} \to \mathbb{R} \quad f(n) = n^2$

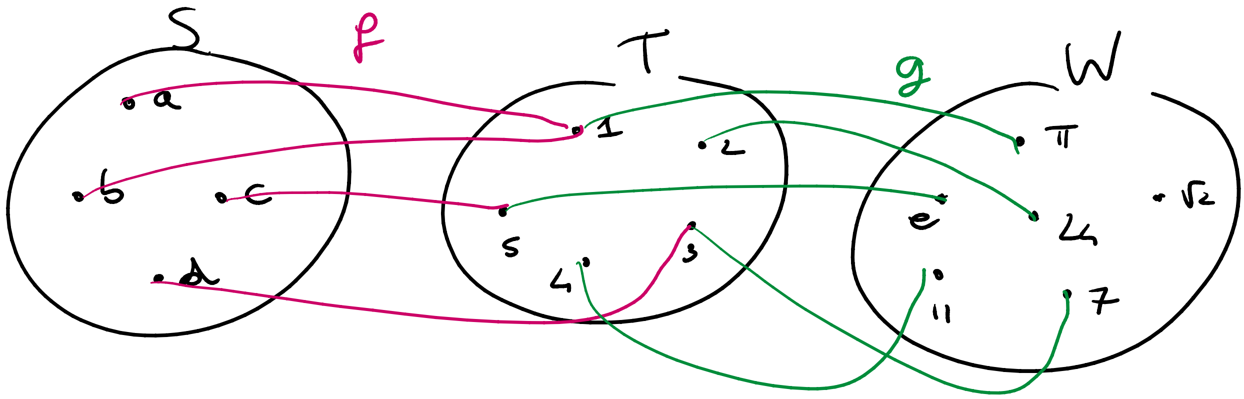
$g : \mathbb{R} \to \mathbb{R} \quad g(x) = x+1$

$$(g \circ f)(n) = g(f(n)) = g(n^2) = n^2+1$$

$$(f \circ g)(x) = f(g(x)) = f(x+1) = (x+1)^2$$

La composizione non è commutativa

OSS - Posso comporre $f: S \to T$ e $g: V \to W$ se, ad esempio $T \subseteq V$, oppure $f(S) \subseteq V$



$g \circ f : S \to W$

$(g \circ f)(a) = g(1) = \pi$

$(g \circ f)(b) = g(1) = \pi$

$(g \circ f)(c) = e$

$(g \circ f)(d) = 7$