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# Convex MV-algebras: Many-valued logics meet decision theory

**Abstract.** This paper introduces a logical analysis of convex combinations within the framework of Łukasiewicz real-valued logic. This provides a natural link between the fields of many-valued logics and decision theory under uncertainty, where the notion of convexity plays a central role. We set out to explore such a link by defining convex operators on MV-algebras, which are the equivalent algebraic semantics of Łukasiewicz logic. This gives us a formal language to reason about the expected value of bounded random variables. As an illustration of the applicability of our framework we present a logical version of the Anscombe-Aumann representation result.

*Keywords*: MV-algebras, convexity, uncertainty measures, Anscombe-Aumann.

## 1. Introduction and motivation

The purpose of this paper is to bring to the foreground an interesting link between algebraic non-classical logics and the representation of (subjective) expected utility in decision theory. To the best of our knowledge this constitutes an as yet unexplored avenue in the rapidly expanding field which puts logical methods to work in the social sciences.

Classical (propositional) logic formalises the notion of "correct deduction" in a framework in which there is one possible state of affairs (or "world") in which every object is evaluated to either "true" or "false". Neither of those features turns out to be particularly useful when modelling rational reasoning. For we often resort to thinking about what the world *might have been* or face situations which are neither definitely true nor false. Modal logics have been developed to capture the first kind of situation whereas reasoning with statements which can be "partially true" is the main motivation behind the field of many-valued logics.

Modal logics have found extensive applications in modelling the epistemic attitudes of rational agents, i.e. in giving mathematically rigorous definitions of what does it mean, for an agent, "to know $\phi$", or "to believe that $\phi$", etc. This had a tremendous impact on the analysis of solution concepts in game theory, notably the logical analysis of the common knowledge assumption

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required by Nash equilibrium. The importance of this application led to a well-established literature linking logic and games.[1]

The idea of extending the expressive power of logical deduction beyond two values is effectively as old as mathematical logic. A first thread of research on this started in the early 1920's and sought to use many-valued logics to investigate purely mathematical questions, including independence in axiomatic systems [3] and the definition of propositional connectives in the light of non-terminating algorithms [5, 28]. More philosophically oriented is the introduction of Łukasiewicz three-valued logic [34], which was motivated by the desire to formalise reasoning with possibly undetermined truth values. This contributed to breaking the Fregean *taboo* which confined the logical valuation of sentences to range exclusively over the binary set. Mathematically then, the idea of letting valuations range over the reals, or more conventiently over the real-unit interval, appeared all to natural. By the early 1960's Łukasiewicz infinite-valued logic [35] had reached its mathematical maturity through (several proofs of) its completeness with respect to a class of algebraic structures, which were to become established by the name of MV-algebras. Since then, the development of the field took place essentially within the bounds of algebraic logic [10, 11, 43]. An important exception is provided by [26] where many-valued logics are applied to the analysis of reasoning with *vagueness*.

This quick historic detour on many-valued logics[2] is useful to appreciate an important difference between the (widespread) applications of the modal extensions of classical logic in economic theory, and the surprisingly little application of their many-valued counterparts. Modal logics found their way in economics and game theory mainly as *meta-linguistic* tools to reason, in a mathematically rigorous way, *about* the epistemic attitudes of rational agents, and the expressive power of the underlying logics. Łukasiewicz logic, on the other hand, was initially motivated by the philosophical reflection on "degrees of truth", but quickly turned into a natural framework to investigate the arithmetic operations over the reals. Compared to classical logic this is a substantial mathematical leap. Indeed, as the 2-element Boolean algebra $2 = (\{0,1\}, \vee, ^*, 0)$ is the semantics for classical propositional logic, the MV-algebra $[0,1]_{MV} = ([0,1], \oplus, ^*, 0)$ provides the standard semantics with respect to which Łukasiewicz logic is sound and complete. Now, the

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[1] The relevant references are too many to mention here. Interested readers may quickly get an impression of the field by consulting the list of publications related to the established conference series LOFT [33] and TARK [49] which have been running uninterruptedly for the past three decades.

[2] We urge the interested reader to consult [24, 36] for more details.

operations on $[0,1]_{MV}$ (see Section 2) define the arithmetic operations of *truncated sum*, *truncated subtraction* and partial order. This leads naturally to ask [12, 39, 40, 14, 31] which notions of *product* can be axiomatised in $[0,1]_{MV}$. Pushing this line of research further, a central contribution of this paper is to argue that Łukasiewicz logic is suitable to accommodate an axiomatisation of *convex combinations*. Since this latter plays a pivotal role in mathematical economics, and in particular decision theory, Łukasiewicz logic is unduly missing in the current interaction between logic and economic theory. [3] Building on recent work by [6, 9, 21] our paper aims at filling this gap.

The paper is organised as follows. Section 2 provides a quick refresher of the main concepts and results on MV-algebras and *states*, intuitively the MV-algebraic counterparts of finitely additive probability measures on Boolean algebras. Convex MV-algebras (CMV-algebras) are defined and investigated in Section 3. Those are MV-algebras endowed with a family of *convexity* operators. Section 3.1 collects some algebraic results aimed at providing a subdirect representation theorem for CMV-algebras (this subsection can be skipped with no conceptual loss by the non-technical reader). A deeper representation of CMV-algebras is presented in Section 4 where we prove a termwise equivalence between CMV-algebras and Riesz MV-algebras – MV-algebras endowed with a scalar product. The main result of this Section can indeed be regarded as a standard completeness theorem, since it shows, among other things, that CMV-algebras share the equational theory of canonical convex combinations on the real unit interval $[0,1]$. This result gives us a powerful tool which we put to work in the proof of our main result which appears in Section 5. Theorem 5.4 effectively shows that states can be described directly in the logical setting of CMV-algebras. Finally Section 6 illustrates how the framework of CMV-algebras paves the way to constructing a formal and conceptual bridge between many-valued logics and decision theory. To this end Theorem 6.5 provides a logico-algebraic formulation of the well-known Anscombe-Aumann Representation. Given the pivotal role of this latter in the development of numerous extensions of the Bayesian framework (see, e.g. [23]), we suspect that this paper is just scratching the surface of a much deeper and exciting new connection between logic and economic theory.

[3] The field of *judgment aggregation* constitutes a bit of an exception to this, with its peculiarly close connection to logic. See e.g. [38] for a general perspective. [16] applies successfully the semantics of Łukasiewicz logic.

## 2. An overview of MV-algebras and states

**Definition 2.1.** An *MV-algebra* is a structure $\mathbf{A} = \langle A, \oplus, ^*, 0\rangle$, where $\oplus$ is a binary operation, $^*$ is a unary operation and 0 is a constant, such that $(A, \oplus, 0)$ is an Abelian monoid and the following axioms are satisfied for every $x, y \in A$:

(i) $(x^*)^* = x$,

(ii) $0^* \oplus x = 0^*$,

(iii) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

The class of MV-algebras forms a variety that we shall denote by $\mathbb{MV}$. We introduce the new constant 1 and three additional operations $\odot$, $\ominus$ and $\to$ as follows:

$$1 = \neg 0, \qquad x \odot y = (x^* \oplus y^*)^*, \qquad x \ominus y = x \odot y^*, \qquad x \to y = x^* \oplus y.$$

The *Chang distance* is the binary operation

$$d(x, y) = (x \ominus y) \oplus (y \ominus x). \tag{I}$$

In the rest of this paper we will always assume that any MV-algebra has at least two elements and thus $0 \neq 1$.

For every MV-algebra $\mathbf{A}$, the binary relation $\leq$ on $A$ given by

$$x \leq y \quad \text{whenever} \quad x \to y = 1$$

is a partial order. In a totally equivalent manner, we can say that $x \leq y$ iff there exists a $z \in A$ such that $x \oplus z = y$ (see [11, Lemma 1.1.2 and 4.2.2]). As a matter of fact, $\leq$ is a lattice order induced by the join $\vee$ and the meet $\wedge$ defined by

$$x \vee y = (x^* \oplus y)^* \oplus y \quad \text{and} \quad x \wedge y = (x^* \vee y^*)^*,$$

respectively. The lattice reduct of $\mathbf{A}$ then becomes a distributive lattice with the top element 1 and the bottom element 0. If the order $\leq$ of $\mathbf{A}$ is total, then $\mathbf{A}$ is said to be an *MV-chain*.

**Example 2.2.** (1) Equip the real unit interval $[0, 1]$ with a binary operation $x \oplus y = \min\{x + y, 1\}$, a unary operation $x^* = 1 - x$ and the constant 0. This structure, that we will denote $[0, 1]_{MV}$, is an MV-chain called the *standard* MV-algebra and Chang's theorem states that it generates the variety $\mathbb{MV}$

both as a variety or a quasi-variety[10, 11]. The further MV-operations behave as follows in $[0,1]_{MV}$: for every $x, y \in [0,1]$, $x \to y = \min\{0, 1-x+y\}$, $x \ominus y = \max\{0, x-y\}$, $d(x,y) = |x-y|$ (the usual Euclidean distance).

(2) Let $X$ be a set and consider the collection $[0,1]^X$ of all functions from $X$ in $[0,1]$. Let us define the operations $\oplus$ and $^*$ pointwise as in (1). Then, if 0 denote the 0-constant function, the algebra $[0,1]^X_{MV} = ([0,1]^X, \oplus, ^*, 0)$ is an MV-algebra. Whenever $X$ is finite, we will call $[0,1]^X_{MV}$ a *finite-dimensional* MV-algebra.

(3) For every $k \in \mathbb{N}$, let $M(k)$ be the set of all continuous and piecewise linear functions from $[0,1]^k$ to $[0,1]$ each piece having integer coefficient. For every $k \in \mathbb{N}$, the MV-algebra $\mathbf{M}(k) = (M(k), \oplus, ^*, 0)$, where $\oplus, ^*$ and 0 are as before is the free MV-algebra over $k$ generators [11]. Notice that $\mathbf{M}(k)$ is an MV-subalgebra of $[0,1]^{[0,1]^k}_{MV}$.

For every MV-algebra $\mathbf{A}$ a subset $I$ of $A$ is an *ideal* if $I$ is downward closed (with respect to the lattice order of $\mathbf{A}$) and $x, y \in I$ implies $x \oplus y \in I$. An ideal is *proper* if it does not coincide with $A$. A proper ideal of $\mathbf{A}$ is said to be *maximal*, provided that so is with respect to the usual set-theoretical inclusion. The set $Max(\mathbf{A})$ of maximal ideals of $\mathbf{A}$ is a compact Hausdorff space with the *spectral* topology, i.e., the topology on $Max(\mathbf{A})$ whose base is constituted by the sets $O_I$ defined as follows: for every ideal $I$ of $\mathbf{A}$, $O_I = \{M \in Max(\mathbf{A}) \mid I \subseteq M\}$.

**Definition 2.3.** An MV-algebra $\mathbf{A}$ is *semisimple* if the intersection of its maximal ideals is $\{0\}$.[4]

The following result represents semisimple MV-algebras as real-valued continuous functions.

**Theorem 2.4.** *Any semisimple MV-algebra* $\mathbf{A}$ *is isomorphic to an MV-algebra of* $[0,1]$*-valued continuous functions defined on the compact Hausdorff space* $Max(\mathbf{A})$.

For every semisimple MV-algebra $\mathbf{A}$ and for every $a \in A$, we denote by $a^*$ its representation as a continuous function from $Max(\mathbf{A})$ to $[0,1]$ given by Theorem 2.4.

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[4] Although every Boolean algebra is semisimple, this is not the case for MV-algebras. We urge the interested reader to consult [11, 13, 43] for an exhaustive treatment of MV-algebras and non-trivial examples of non-semisimple MV-algebras.

States of MV-algebras were introduced by Mundici in [42] as averaging values of Łukasiewicz truth-valuations. For the purposes of this paper, it is important to anticipate that the main result we will recall in this section, Theorem 2.7, presents states as expected values of bounded random variables.

**Definition 2.5.** A *state* of an MV-algebra $\mathbf{A}$ is a map $s : A \to [0, 1]$ satisfying the following conditions

(1) $s(1) = 1$,

(2) for all $x, y \in A$ such that $x \odot y = 0$, $s(x \oplus y) = s(x) + s(y)$.

A state of $\mathbf{A}$ is said to be *faithful* if $s(x) = 0$, implies $x = 0$.

While condition (1) says that every state is *normalized*, (2) is usually called *additivity* with respect to Łukasiewicz sum $\oplus$. Indeed, the requirement $x \odot y = 0$ is analogous to disjointness of a pair of elements in a Boolean algebra. Indeed, if $\mathbf{A}$ is a Boolean algebra, then $x \odot y = 0$ iff $x \wedge y = 0$. Thus states provides a generalizations of finitely additive probabilities to the realm of MV-algebras: every finitely additive probability on a Boolean algebra is a state as a special case of the above definition. In particular, every Borel probability measure is a state as well. The following examples make this point clear.

**Example 2.6.** (1) Any Boolean algebra $\mathbf{B}$ is an MV-algebra in which the MV-operations $\oplus$ and $\odot$ coincide with the lattice operations $\vee$ and $\wedge$, respectively. Every state $s$ of $\mathbf{B}$ is a finitely additive probability since the condition (2) reads as follows:

$$\text{if } a \wedge b = 0, \text{ then } s(a \vee b) = s(a) + s(b).$$

(2) Every homomorphism $h$ of an MV-algebra $\mathbf{A}$ into the standard MV-algebra $[0, 1]_{MV}$ is a state of $\mathbf{A}$. In particular, whenever $\mathbf{A}$ is a subalgebra of the MV-algebra $[0, 1]^X$ of all functions $X \to [0, 1]$. For any $x \in X$ the evaluation mapping $s_x \colon A \to [0, 1]$ given by

$$s_x(f) = f(x), \quad f \in A,$$

is a state of $\mathbf{A}$.

(3) Let $\mathbf{A}$ be a finite-dimensional MV-algebra (recall Example 2.2 (2)). It is well known that, if $k = |X|$, than $\mathbf{A}$ has exactly $k$ homomorphisms

in $[0,1]_{MV}$, namely the projection maps $\pi_i : \mathbf{A} \to [0,1]_{MV}$. For every $\lambda_1, \ldots, \lambda_k \in [0,1]$ such that $\sum_{i=1}^k \lambda_i = 1$, the map $s : A \to [0,1]$ defined by

$$s(f) = \sum_{i=1}^{k} \lambda_i \cdot \pi_i(f) \tag{II}$$

is a state of $\mathbf{A}$. Indeed, as the next theorem shows, every state of $\mathbf{A}$ arises in this way, that is, states of finite dimensional MV-algebras are the same as convex combinations of the projection maps of $\mathbf{A}$ into $[0,1]_{MV}$. Moreover, $s$ is faithful iff $\lambda_i > 0$ for every $i = 1, \ldots, k$.

The following theorem, independently proved by Kroupa [30] and Panti [46] (see also [19, §4]), is a generalization of the above Example 2.6 (3). In what follows, we will denote by $\mathcal{B}(Max(\mathbf{A}))$ the Borel $\sigma$-algebra of the compact Hausdorff space $Max(\mathbf{A})$ of maximal ideals of an MV-algebra $\mathbf{A}$.

**Theorem 2.7.** *For every semisimple MV-algebra* $\mathbf{A}$ *and for every state* $s$ *of* $\mathbf{A}$ *there exists a unique regular, Borel probability measure* $\mu$ *on* $\mathcal{B}(Max(\mathbf{A}))$ *such that, for every* $a \in A$,

$$s(a) = \int_{Max(\mathbf{A})} a^* \, \mathrm{d}\mu.$$

**Remark 2.8** (Continuous random variables)**.** The representation theorem for semisimple MV-algebra, Theorem 2.4, is the first ingredient towards an interpretation of MV-algebras as algebras of *continuous bounded random variables*, that is, up to renormalization, continuous $[0,1]$-valued random variables.

States provide the second ingredient. Indeed, to every semisimple MV-algebra $\mathbf{A}$ and every state $s$ of $\mathbf{A}$, we define a measure space $(\Sigma_{\mathbf{A}}, \mathcal{F}_{\mathbf{A}}, \mu_s)$, where $\Sigma_{\mathbf{A}} = Max(\mathbf{A})$, $\mathcal{F}_{\mathbf{A}} = \mathcal{B}(Max(\mathbf{A}))$ and $\mu_s$ is that unique Borel regular measure over $\mathcal{B}(Max(\mathbf{A}))$ as given in Theorem 2.7. In other words, for any MV-algebra $\mathbf{A}$ and for any state $s$, there exists a unique measure space such that $s(a)$ is the *expected value* of $a^*$.[5]

[5] Following [44], an algebra of random variables contains the constants and it is closed for sum and product. Since we focus on bounded random variables, the truncated sum of an MV-algebra is the natural replacement for the regular sum. Moreover, being our structures essentially algebras of $[0,1]$-valued functions, they can be endowed with a structure of PMV-algebra. However, since our focus is on additivity – and not independence, which depends on the definition of a binary product – this treatment goes beyond the focus of the present paper. Nonetheless, [31] provides a suitable framework for a more general treatment of bounded random variables. Indeed in [31] $f$MV-algebras are defined as MV-algebras endowed both scalar product and binary product.

### 2.1. MV-algebras with a product conjunction

The language of MV-algebras can be expanded by means of product-like operations. In particular, it is possible to define a binary internal product giving rise to a class of algebras known as *PMV-algebras* or a scalar multiplication so defining the class of *Riesz MV-algebras*.

In the following definitions we will use the notion of partial sum + introduced and further investigated by Dvurečenskij in [17, 18]. Formally, let $\mathbf{A}$ be an MV-algebra and $x, y$ elements of $A$. Then $x+y$ is defined iff $x \odot y = 0$ and in this case we have $x + y = x \oplus y$.

**Definition 2.9.** A PMV-algebra is a structure $\mathbf{P} = (P, \oplus, ^*, \cdot, 0)$ such that $(P, \oplus, ^*, 0)$ is an MV-algebra, and $\cdot : P \times P \to P$ satisfies the following, for any $x, y, z \in P$

(P1) If $x + y$ is defined, so is $z \cdot x + z \cdot y$ and it coincides with $z \cdot (x + y)$,

(P2) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,

(P3) $1 \cdot x = x \cdot 1 = x$,

(P4) $x \cdot y = y \cdot x$.

In [12, 39] it is shown that PMV-algebras form a variety, denoted by $\mathbb{PMV}$. PMV-algebras can be defined more in general, in the not-unital and not-commutative case [12].

The algebra $[0,1]_{PMV} = ([0,1]_{MV}, \cdot)$ is a PMV-algebra when the operation $\cdot$ is the usual multiplication between real numbers. The algebra $[0,1]_{PMV}$ generates a proper sub-variety of $\mathbb{PMV}$ (see [27] for details).

PMV$^+$-algebras are the objects in the sub-quasivariety of $\mathbb{PMV}$ axiomatized by $x^2 = 0 \Rightarrow x = 0$. The quasivariety $\mathbb{PMV}^+$ of PMV$^+$-algebras is generated by $[0,1]_{PMV}$ [40].

**Definition 2.10** ([14])**.** A *Riesz MV-algebra* is a pair $(\mathbf{A}, \mathscr{R})$ where $\mathbf{A}$ is an MV-algebra and $\mathscr{R} = \{\alpha(\cdot)\}_{\alpha \in [0,1]}$ is a family of unary operators on $A$ such that the following hold, for any $x, y \in A$ and $\alpha, \beta \in [0,1]$.

(R1) If $\alpha + \beta$ is defined, so is $\alpha x + \beta x$ and it coincides with $(\alpha + \beta)x$,

(R2) If $x + y$ is defined, so is $\alpha x + \alpha y$ and it coincides with $\alpha(x + y)$,

(R3) $\alpha(\beta x) = (\alpha \cdot \beta)x$,

(R4) $1x = x$.

The standard Riesz MV-algebra is $[0,1]_{RMV} = ([0,1]_{MV}, \{\alpha\}_{\alpha \in [0,1]})$, where $\alpha(x) = \alpha x$ is the usual multiplication between real numbers. Indeed, $[0,1]_{RMV} = [0,1]_{PMV}$.

Analogously to the case of PMV-algebras, it is possible to show that Riesz MV-algebras form a variety denoted by $\mathbb{RMV}$ [14, Theorem 2]. In the rest of this paper we will quite frequently adopt the following notation. If $\{\mathbf{A}_1, \ldots, \mathbf{A}_t\}$ is a (not necessarily finite) set of algebras of the same type, $\mathbb{V}(\{\mathbf{A}_1, \ldots, \mathbf{A}_t\})$ denotes the variety generated by $\{\mathbf{A}_1, \ldots, \mathbf{A}_t\}$ (see [7, Definition 9.4]). For the sake of a lighter notation, we will write $\mathbb{V}(\mathbf{A})$ instead of $\mathbb{V}(\{\mathbf{A}\})$.

**Theorem 2.11.** *Every Riesz MV-algebra is subdirect product of totally ordered Riesz MV-algebras. Furthermore,*

$$\mathbb{RMV} = \mathbb{V}([0,1]_{RMV}) = \mathbb{V}([0,1]_{PMV}).$$

**Remark 2.12** (A categorical perspective)**.** A *unital lattice-ordered Abelian group* (*$\ell u$-group* for short) is a lattice-ordered Abelian group $\mathbf{G}$ equipped with a constant $u$ such that for every $x \in G$ there is a natural number $n$ such that $x \leq nu$. Such constant $u$ is called strong unit.

Given an $\ell u$-group $\mathbf{G}$, the algebra $\mathbf{A} = (A, \oplus, ^*, 0)$ where $A = \{x \in G : 0 \leq x \leq u\}$, $x^* = u - x$ and $x \oplus y = (x + y) \wedge u$ is an MV-algebra, which will be denoted by $\Gamma(\mathbf{G}, u)$. For every morphism $h$ (i.e., a group homomorphism preserving the strong unit) between the $\ell u$-groups $(\mathbf{G}, u)$ and $(\mathbf{G}', u')$ we denote by $\Gamma(h)$ its restriction to $\Gamma(\mathbf{G}, u)$. We hence obtain a functor from the category of $\ell u$-groups into the category of MV-algebras. By an important result by Mundici [41], $\Gamma$ has an adjoint $\Gamma^{-1}$ and the pair $(\Gamma, \Gamma^{-1})$ constitutes an equivalence of categories. The categorical equivalence between MV-algebras and $\ell u$-groups extends in the same fashion to PMV-algebras and a subclass of lattice-ordered *rings* with strong unit [12] and Riesz MV-algebras and *Riesz spaces* [14] (i.e., vector lattices) with strong unit.

## 3. Convex MV-algebras

An axiomatic approach to convex combinations has been recently investigated by Fritz, Brown and Capraro [6, 9, 21] who explore convex spaces in terms of a family of binary operations satisfying certain compatibility conditions. In this section we will develop a similar approach within the framework of MV-algebras.

**Definition 3.1.** An *MV-algebra with convexity operators* (*CMV-algebra* for short) is an MV-algebra $\mathbf{A}$ together with a family of binary operators $\mathscr{C} = \{cc_\alpha\}_{\alpha \in [0,1]}$ satisfying the following properties for every $x, y, z \in A$, $\alpha, \beta \in [0,1]$:

(C1) $cc_0(x, y) = y$;

(C2) $cc_\alpha(x, y) = cc_{1-\alpha}(y, x)$;

(C3) $cc_\alpha(x, x) = x$;

(C4) $cc_\alpha(cc_\beta(x, y), z) = cc_{\alpha\beta}(x, cc_\gamma(y, z))$, with $\gamma$ arbitrary if $\alpha = \beta = 1$ and $\gamma = \frac{\alpha(1-\beta)}{1-\alpha\beta}$ otherwise;

(C5) For all $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, $cc_\alpha(x, 0) + cc_\beta(x, 0)$ is defined and it coincides with $cc_{\alpha+\beta}(x, 0)$;

(C6) If $x+x'$ and $y+y'$ are defined, so is $cc_\alpha(x, y)+cc_\alpha(x', y')$ and it coincides with $cc_\alpha(x + x', y + y')$;

(C7) $cc_\alpha(x, y)^* = cc_\alpha(x^*, y^*)$.

Obviously, CMV-algebras form a quasi-variety denoted by $\mathbb{CMV}$.

**Example 3.2.** (i) Let $[0, 1]_{PMV}$ be the standard PMV-algebra and let, for any $\alpha \in [0, 1]$, $cc_\alpha$ be defined as follows:

$$\text{for all } x, y \in [0, 1],\ cc_\alpha(x, y) = \alpha x \oplus (1 - \alpha)y.$$

By definition of the operations on $[0, 1]$ it follows that

$$\alpha x + (1 - \alpha)y \leq \alpha(x \vee y) + (1 - \alpha)(x \vee y) = x \vee y \leq 1.$$

Hence, it is possible to write the operators $cc_\alpha$ as

$$cc_\alpha(x, y) = \alpha x + (1 - \alpha)y.$$

Let us denote $\mathscr{C} = \{cc_\alpha\}_{\alpha\in[0,1]}$ and let $[0, 1]_{CMV} = ([0, 1]_{MV}, \mathscr{C})$.

It is not difficult to show that $[0, 1]_{CMV}$ is a CMV-algebra. Let us prove (C6); the proof of the remaining equations is left to the reader. Assume that $x, x', y, y' \in [0, 1]$ and $x + x'$ and $y + y'$ are defined. Then $cc_\alpha(x, y) + cc_\alpha(x', y') = (\alpha x+(1-\alpha)y)+(\alpha x'+(1-\alpha y')) = \alpha(x+x')+(1-\alpha)(y+y') \leq (x+x') \vee (y+y')$. In turn, since $x+x'$ and $y+y'$ are defined, so is $(x+x') \vee (y + y')$ and hence $cc_\alpha(x, y) + cc_\alpha(x', y') \leq 1$. Further notice that, being the partial sum associative, $cc_\alpha(x, y) + cc_\alpha(x', y') = \alpha(x + x') + (1 - \alpha)(y + y')$ and hence $cc_\alpha(x, y) + cc_\alpha(x', y') = cc_\alpha(x + x', y + y')$.

The CMV-algebra $[0, 1]_{CMV} = ([0, 1]_{MV}, \mathscr{C})$ will be henceforth called the *standard* CMV-algebra.

(ii) Every Riesz MV-algebra **A** can be endowed with a CMV-structure defining, for every $x, y \in A$ and for every $\alpha \in [0, 1]$, $cc_\alpha(x, y) = \alpha x\oplus(1-\alpha)y$. Indeed, as we already observed in (i) and since $[0, 1]_{PMV}$ generates $\mathbb{RMV}$ as a variety, for every Riesz MV-algebra **A**, every $x, y \in A$ and every $\alpha \in [0, 1]$, we have $cc_\alpha(x, y) = \alpha x \oplus (1 - \alpha)y = \alpha x + (1 - \alpha)y$. As we will show in Section 4, this is the most general example of CMV-algebra.

In the following proposition we prove some basic properties regarding the behaviour of the operators $cc_\alpha$, with respect to manipulations of the $\alpha$'s.

**Proposition 3.3.** *Let* $(\mathbf{A}, \mathscr{C})$ *be CMV-algebra. Then, for all* $x, y, z, a, b \in A$ *and for all* $\alpha, \beta, \gamma \in [0, 1]$*, the following hold:*

*(i)* $cc_1(x, y) = x$;

*(ii)* $cc_\alpha(cc_\beta(x, y), y) = cc_{\alpha\beta}(x, y)$;

*(iii)* $cc_\alpha(cc_\beta(x, z), cc_\gamma(y, z)) = cc_\mu(cc_\nu(x, y), z)$, *with* $\mu = \alpha\beta + (1 - \alpha)\gamma$, $\nu = \frac{\alpha\beta}{\mu}$ *and* $\alpha, \beta, \gamma \in (0, 1)$;

*(iv)* $cc_\alpha(cc_\beta(a, b), cc_\beta(x, y)) = cc_\beta(cc_\alpha(a, x), cc_\alpha(b, y))$, *with* $\alpha, \beta \in (0, 1)$;

*(v) If* $\alpha \le \beta$*, then* $cc_\alpha(x, 0) \le cc_\beta(x, 0)$.

*Proof.* See Appendix. □

The following two propositions collect several properties concerning the algebraic interplay between convex combinations and MV-operations.

**Proposition 3.4.** *In every CMV-algebra* $(\mathbf{A}, \mathscr{C})$ *the following hold for every* $x, x', y, y', z \in A$ *and* $\alpha \in [0, 1]$.

*(i) If* $x \le x'$*, then* $cc_\alpha(x, y) \le cc_\alpha(x', y)$,

*(ii) If* $y \le y'$*, then* $cc_\alpha(x, y) \le cc_\alpha(x, y')$,

*(iii)* $cc_\alpha(x, 0) \le x$ *and* $cc_\alpha(0, y) \le y$,

*(iv)* $x \wedge y \le cc_\alpha(x, y) \le x \vee y$,

*(v)* $x \odot y \le cc_\alpha(x, y) \le x \oplus y$,

*(vi)* $cc_\alpha(x, y) \vee cc_\alpha(x', y') \le cc_\alpha(x \vee x', y \vee y')$,

*(vii)* $cc_\alpha(x, y) \wedge cc_\alpha(x', y') \ge cc_\alpha(x \wedge x', y \wedge y')$,

*(viii) If* $x \odot z = y \odot z = 0$*, then* $cc_\alpha(x, y) \odot z = 0$ *and* $cc_\alpha(x \oplus z, y \oplus z) = cc_\alpha(x, y) \oplus z$.

*(ix)* $cc_\alpha(x^*, 0) = cc_\alpha(x, 0)^* \odot cc_\alpha(1, 0)$,

*(x)* $cc_\alpha(x, 0) \odot cc_\alpha(y, 0)^* \le cc_\alpha(x \odot y^*, 0)$,

*(xi)* $d(cc_\alpha(x, 0), cc_\alpha(y, 0)) \le cc_\alpha(d(x, y), 0)$ *and* $d(cc_\alpha(0, x), cc_\alpha(0, y)) \le cc_\alpha(0, d(x, y))$

*Proof.* See Appendix. □

**Proposition 3.5.** *The following hold in every CMV-algebra* $(\mathbf{A}, \mathscr{C})$.

*(i) If* $x > 0$ *and* $\alpha > 0$*, then* $cc_\alpha(x, 0) > 0$

*(ii)* *If $x < x'$, then $cc_\alpha(x, y) < cc_\alpha(x', y)$ for all $y \in A$ and $\alpha \in (0, 1]$.*

*(iii)* *If $x, y$ are incomparable, then, for every $\alpha$, $x \wedge y < cc_\alpha(x, y) < x \vee y$ for every $\alpha \in [0, 1]$.*

*Proof.* See Appendix. □

**Remark 3.6.** As we already recalled in the Introduction, the notion of *convexity* has been widely studied in the literature from several different perspectives. In particular, we would like to briefly focus on the way *convex structures* have been axiomatized by van de Vel in [50]: Given a non-empty set $X$, a subset $\mathcal{C}$ of its powerset is called a *convexity on* $X$ if:

(CS1) The empty set $\emptyset$ and $X$ are in $\mathcal{C}$,

(CS2) $\mathcal{C}$ is closed under intersections and nested unions.

The pair $(X, \mathcal{C})$ is hence called a *convex structure* in [50]. The following examples show analogies and differences between van de Vel's approach and the logico-algebraic perspective we pushed forward in our axiomatization.

Let $[0, 1]_{CMV} = ([0, 1]_{MV}, \mathscr{C})$ be the CMV-algebra of Example 3.2 (i). For every finite subset $X$ of $[0, 1]$, let

$$CC(X) = \{cc_\alpha(x_1, x_2) \mid x_1, x_2 \in X, \alpha \in [0, 1]\}$$

and call

$$\mathcal{C} = \{CC(X) \mid X \subset A, X \text{ finite}\} \cup \{[0, 1]\}.$$

Then $([0, 1], \mathcal{C})$ is a convex structure. Indeed, $\emptyset$ and $[0, 1]$ trivially belong to $\mathcal{C}$. Moreover, let $\mathcal{D}$ be is a collection of elements of $\mathcal{C}$. If $\bigcap \mathcal{D} = \emptyset$, it belongs to $\mathcal{C}$ by definition, otherwise we have

$$\bigcap \mathcal{D} = CC\left(\left\{\inf_{Y \in \mathcal{D}}\{\max Y\}, \sup_{Y \in \mathcal{D}}\{\min Y\}\right\}\right) \in \mathcal{C}.$$

In a similar way one can check that if $\mathcal{D}$ is a family of nested subsets of $\mathcal{C}$, its union belongs to $\mathcal{C}$.

In the same book, van de Vel introduces a notion of *lattice convexity* which is essentially definied in this way: let $(L, \wedge, \vee)$ be a lattice and let $F$ be a finite subset of $L$. Then, the *lattice convexity* of $F$ is defined as $\text{co}(F) = \{x \in L \mid \inf(F) \leq x \leq \sup(F)\}$. Since CMV-algebras have a lattice reduct, one may wonder if lattice convexity could be handled in our setting. Although the set $\text{co}(F)$ could be obviously defined in every CMV-algebra, the sets $CC(X)$ definied above do not capture lattice convexity in general.

For instance, consider a nontrivial ultrapower $^*[0,1]_{CMV}$ of the standard CMV-algebra $[0,1]_{CMV}$ and let $F = \{0,1\}$. Then, co$(F)$ clearly coincides with $^*[0,1]$, but on the other hand, $CC(F) = [0,1]$. (In particular, if $\varepsilon$ is infinitesimal, there is no way to express $\varepsilon$ as $cc_\alpha(0,1)$, since $\alpha \in [0,1]$). Therefore, $CC(F) \subseteq$ co$(F)$ and the inclusion is proper. The situation is yet more evident if the CMV-algebra is not totally ordered. Consider in fact a subset $F$ of a CMV-algebra $(\mathbf{A}, \mathscr{C})$ such that $\sup(F) \notin F$. Then, although $\sup(F) \in$ co$(F)$, there is no apparent way to express it as $cc_\alpha(x,y)$ for some $x, y \in F$ and $\alpha \in [0,1]$. In specific terms, consider a CMV-algebra whose Boolean skeleton is the 4-element Boolean algebra and let $F = \{a, a^*\}$ the set of its atoms. Then $a \vee a^* = 1 \in$ co$(F)$. On the other hand, for every $\alpha \in [0,1]$, $cc_\alpha(a, a^*) < 1$ (recall Proposition 3.5 (iii)), whence again $CC(F) \subseteq$ co$(F)$ and the inclusion is proper.

This remark seems to revel that the notion of convexity captured by CMV-algebras is slightly more specific than that one axiomatized in [50], the latter being more order-theoretic, while the former being more constraint by its algebraic nature. A deepening on this subject will be part of our future work.

### 3.1. Convex ideals and subdirectly irreducible CMV-algebras

A subset $I$ of a CMV-algebra $(\mathbf{A}, \mathscr{C})$ is a $\mathscr{C}$-ideal provided that $0 \in I$, $x, y \in I$ implies $x \oplus y \in I$, $I$ is downward closed (i.e., it is an MV-ideal), and $x, y \in I$, implies $cc_\alpha(x,y) \in I$ for every $cc_\alpha \in \mathscr{C}$. From Proposition 3.4 (v), for every $x, y \in A$ and for every $\alpha \in [0,1]$, $cc_\alpha(x,y) \leq x \oplus y$ and hence every MV-ideal of $\mathbf{A}$ is a $\mathscr{C}$-ideal of $(\mathbf{A}, \mathscr{C})$. Thus, the $\mathscr{C}$-ideals of $(\mathbf{A}, \mathscr{C})$ coincide with the MV-ideals of $\mathbf{A}$. Since the lattice $(\mathcal{I}(\mathbf{A}), \subseteq)$ of MV-ideals of any MV-algebra $\mathbf{A}$ is isomorphic with the lattice $(Con(\mathbf{A}), \subseteq)$ of the congruences of $\mathbf{A}$ [11, Proposition 1.2.6], the lattice $(\mathcal{I}_{\mathscr{C}}(\mathbf{A}), \subseteq)$ of $\mathscr{C}$-ideals of a CMV-algebra $(\mathbf{A}, \mathscr{C})$ is isomorphic with the lattice of congruence of its MV-reduct.

**Lemma 3.7.** *Let $I$ be an ideal for a CMV-algebra $(\mathbf{A}, \mathscr{C})$, and let $x, y, s, t$ be elements in $A$ such that $d(x,y) \in I$ and $d(s,t) \in I$. Then $d(cc_\alpha(x,s), cc_\alpha(y,t)) \in I$*

*Proof.* By [11, Propostion 1.2.5(v)],

$$d(cc_\alpha(x,s), cc_\alpha(y,t)) = d(cc_\alpha(x,0) + cc_\alpha(0,s), cc_\alpha(y,0) + cc_\alpha(0,t)) \leq$$
$$\leq d(cc_\alpha(x,0), cc_\alpha(y,0)) + d(cc_\alpha(0,s), cc_\alpha(0,t))$$

Hence, by Proposition 3.4 (xi),

$d(cc_\alpha(x,0), cc_\alpha(y,0)) + d(cc_\alpha(0,s), cc_\alpha(0,t)) \leq cc_\alpha(d(x,y),0) + cc_\alpha(0,d(s,t))$.

Thus we have,

$$d(cc_\alpha(x,s), cc_\alpha(y,t)) \leq cc_\alpha(d(x,y),0) + cc_\alpha(0,d(s,t)). \tag{III}$$

By hypothesis and Proposition 3.4 (iv) we get $cc_\alpha(d(x,y),0) \leq d(x,y) \in I$ and $cc_\alpha(0,d(s,t)) \leq d(s,t) \in I$, whence $cc_\alpha(d(x,y),0) \in I$, $cc_\alpha(0,d(s,t)) \in I$ and, since ideals are closed under $\oplus$, $cc_\alpha(d(x,y),0) + cc_\alpha(0,d(s,t)) \in I$. Therefore, (III) settles the claim. □

**Theorem 3.8.** *Every CMV-algebra $(\mathbf{A}, \mathscr{C})$ and its MV-reduct $\mathbf{A}$ have the same congruences. In particular $(\mathbf{A}, \mathscr{C})$ is subdirectly irreducible iff so is $\mathbf{A}$.*

*Proof.* Let $I$ be an ideal of $(\mathbf{A}, \mathscr{C})$. For every $x, y \in A$, we define

$$x\theta_I y \text{ if and only if } d(x,y) \in I.$$

Since $I$ is in particular an MV-ideal, $\theta_I$ is a congruence for the MV-algebra $\mathbf{A}$ [11, Proposition 1.2.6]. Hence, $\theta_I$ is a congruence of $(\mathbf{A}, \mathscr{C})$ if $x\theta_I y$ and $s\theta_I t$ imply $cc_\alpha(x,s)\theta_I cc_\alpha(y,t)$. This follows by Lemma 3.7 since, by definition of $\theta_I$, both $d(x,y)$ and $d(s,t)$ belongs to $I$ and hence so does $d(cc_\alpha(x,s), cc_\alpha(y,t))$, that is $cc_\alpha(x,s)\theta_I cc_\alpha(y,t)$.

Conversely, if $\theta$ is a congruence of $(\mathbf{A}, \mathscr{C})$, we define $I_\theta$ as $\{x \in A \mid x\theta 0\}$. By [11, Proposition 1.2.6], $I_\theta$ is an MV-ideal, hence it is a convex ideal.

Hence $(\mathcal{I}_{\mathscr{C}}(\mathbf{A}), \subseteq)$ and $(Con_{\mathscr{C}}(\mathbf{A}), \subseteq)$ are isomorphic and from what we stated above about convex ideal and MV-congruences, it immediately follows that $(\mathbf{A}, \mathscr{C})$ and its MV-reduct $\mathbf{A}$ have the same congruences.

Finally, since being subdirectly irreducible only depends of the lattice of congruences, $(\mathbf{A}, \mathscr{C})$ is subdirectly irreducible iff so is $\mathbf{A}$. □

**Corollary 3.9.** *Any subdirectly irreducible CMV-algebra is totally ordered and hence every CMV-algebra is subdirect product of totally ordered CMV-algebras.*

*Proof.* The claim is a straightforward consequence of [4, Chapter VIII, Theorem 15], Theorem 3.8 and the fact that any subdirectly irreducible MV-algebra is totally ordered. □

## 4. Termwise equivalence: A standard completeness theorem

In this section we will show a termwise equivalence between CMV-algebras and Riesz MV-algebras. Afterwards, we will discuss some consequences of this equivalence.

Let us consider two maps $\mathcal{R} : \mathbb{CMV} \to \mathbb{RMV}$ and $\mathcal{C} : \mathbb{RMV} \to \mathbb{CMV}$ defined in the following manner:

- For every $(\mathbf{A}, \mathscr{C}) \in \mathbb{CMV}$, let $\mathcal{R}(\mathbf{A}, \mathscr{C}) = (\mathbf{A}, \mathscr{R})$ where, for every $\alpha \in [0, 1]$ and for every $x \in A$,
$$\alpha(x) = cc_\alpha(x, 0). \tag{IV}$$
- For every $(\mathbf{A}, \mathscr{R}) \in \mathbb{RMV}$, let $\mathcal{C}(\mathbf{A}, \mathscr{R}) = (\mathbf{A}, \mathscr{C})$ where for every $\alpha \in [0, 1]$ and $x, y \in A$,
$$cc_\alpha(x, y) = \alpha x \oplus (1 - \alpha)y. \tag{V}$$

In order to establish the termwise equivalence between CMV-algebras and Riesz MV-algebras, we must prove the following: for any $(\mathbf{A}, \mathscr{C}) \in \mathbb{CMV}$, $\mathcal{C}(\mathcal{R}(\mathbf{A}, \mathscr{C}))$ is isomorphic with $(\mathbf{A}, \mathscr{C})$ (in symbols, $\mathcal{C}(\mathcal{R}(\mathbf{A}, \mathscr{C})) \cong (\mathbf{A}, \mathscr{C})$) and for any $(\mathbf{A}, \mathscr{R}) \in \mathbb{RMV}$, $\mathcal{R}(\mathcal{C}(\mathbf{A}, \mathscr{R})) \cong (\mathbf{A}, \mathscr{R})$.

**Lemma 4.1.** *Let $(\mathbf{A}, \mathscr{C})$ be a CMV-algebra. Then $\mathcal{R}(\mathbf{A}, \mathscr{C})$ is a Riesz MV-algebra.*

*Proof.* We are going to show that, for every CMV-algebra $(\mathbf{A}, \mathscr{C})$, $\mathcal{R}(\mathbf{A}, \mathscr{C}) = (\mathbf{A}, \mathscr{R})$ satisfies the conditions of Definition 2.10.

1. If $\alpha + \beta$ is defined in $[0, 1]_{PMV}$, for every $x \in A$, $\alpha x + \beta y = cc_\alpha(x, 0) + cc_\beta(x, 0)$. Then the claim directly follows from (C5).
2. Assume $x + y$ is defined in $\mathbf{A}$, then, for every $\alpha \in [0, 1]$, $\alpha x + \alpha y = cc_\alpha(x, 0) + cc_\alpha(y, 0)$. From (C6) and the fact that $x+y$ is defined, it follows that $\alpha x+\alpha y = cc_\alpha(x, 0)+cc_\alpha(y, 0)$ is defined and $\alpha x+\alpha y = cc_\alpha(x+y, 0)$, i.e. $\alpha(x + y)$.
3. By Proposition 3.3 (ii), $(\alpha\beta) = \alpha \circ \beta$, where $\circ$ denotes the composition map. Indeed $(\alpha\beta)(x) = cc_{\alpha\beta}(x, 0) = cc_\alpha(cc_\beta(x, 0)) = \alpha(\beta(x)) = \alpha\circ\beta(x)$.
4. $1(x) = cc_1(x, 0) = x$ by Proposition 3.3 (i).

Hence our claim is settled.

□

**Lemma 4.2.** *Let $(\mathbf{A}, \mathscr{R})$ be a Riesz MV-algebra. Then $\mathcal{C}(\mathbf{A}, \mathscr{R})$ is a CMV-algebra.*

*Proof.* Since $[0, 1]_{PMV}$ generates the variety $\mathbb{RMV}$ of Riesz MV-algebras, the operators $cc_\alpha(x, y)$ defined as in equation (V) satisfy $cc_\alpha(x, y) \leq x \vee y$ (see Example 3.2). Therefore,
$$cc_\alpha(x, y) = \alpha x \oplus (1 - \alpha)y = \alpha x + (1 - \alpha)y. \tag{VI}$$

It is not difficult to prove that $(\mathbf{A}, \mathscr{C}) = \mathcal{C}(\mathbf{A}, \mathscr{R})$, where the operators in $\mathscr{C}$ are as in (VI) is a CMV-algebra.

Indeed, (C1)–(C3) and (C7) can be easily proved by direct computation. (C5) and (C6) are analogous to (R1) and (R2) of Definition 2.10. Let hence prove (C4).

If $\alpha = \beta = 1$, then the claim follows by Proposition 3.3 (i). Let hence assume that $\alpha < 1$ without loss of generality. Then, $cc_\alpha(cc_\beta(x, y), z) = \alpha(\beta x + (1 - \beta y)) + (1 - \alpha)z = [\alpha\beta x + \alpha(1 - \beta)y] + (1 - \alpha)z$.
On the other hand, let $\gamma = \frac{\alpha(1-\beta)}{1-\alpha\beta}$. Hence $1 - \gamma = \frac{1-\alpha}{1-\alpha\beta}$. Then

$$cc_{\alpha\beta}(x, cc_\gamma(y, z)) = \alpha\beta x + (1 - \alpha\beta)\left(\frac{\alpha(1 - \beta)}{1 - \alpha\beta} y + \frac{1 - \alpha}{1 - \alpha\beta} z\right).$$

By (R1) and (R3) of Definition 2.10, the latter equals to

$$cc_{\alpha\beta}(x, cc_\gamma(y, z)) = \alpha\beta x + [\alpha(1 - \beta)y + (1 - \alpha)z]\,,$$

and the claim follows by associativity of $+$. □

**Theorem 4.3.** *CMV-algebras and Riesz MV-algebras are termwise equivalent.*

*Proof.* From Lemmas 4.1 and 4.2, it suffices to prove what follows:

(i) for any $(\mathbf{A}, \mathscr{C}) \in \mathbb{CMV}$, let $\mathcal{C}(\mathcal{R}(\mathbf{A}, \mathscr{C})) = (\mathbf{A}', \mathscr{C}')$. Then, $(\mathbf{A}', \mathscr{C}') \cong (\mathbf{A}, \mathscr{C})$;

(ii) for any $(\mathbf{A}, \mathscr{R}) \in \mathbb{RMV}$, let $\mathcal{R}(\mathcal{C}(\mathbf{A}, \mathscr{R})) = (\mathbf{A}', \mathscr{R}')$. Then, $(\mathbf{A}', \mathscr{R}') \cong (\mathbf{A}, \mathscr{R})$.

As for (i), for every $\alpha \in [0, 1]$ and $x, y \in A$, Equations (IV-VI) give

$$cc'_\alpha(x, y) = \alpha' x + (1 - \alpha)' y = cc_\alpha(x, 0) + cc_{1-\alpha}(y, 0) = cc_\alpha(x, 0) + cc_\alpha(0, y).$$

By a direct inspection on the proof of Proposition 3.4 (ix), the latter equals $cc_\alpha(x, y)$.

In order to prove (ii), for all $\alpha \in [0, 1]$ and $x \in A$, equations (IV-VI) ensure that

$$\alpha' x = cc'_\alpha(x, 0) = \alpha x + (1 - \alpha)0 = \alpha x.$$

Hence the claim is settled. □

Theorem 4.3 has several relevant consequences that we are now going to present. First of all, since the class of Riesz MV-algebras forms a variety, and varieties are preserved by termwise equivalences, $\mathbb{CMV}$ is a variety as well. Moreover, the following holds.

**Corollary 4.4.** $\mathbb{CMV} = \mathbb{V}([0,1]_{CMV})$.

*Proof.* A direct inspection on the proof of Theorem 4.3 shows that the CMV-algebra $\mathcal{C}([0,1]_{PMV})$ is isomorphic to $[0,1]_{CMV}$ (remind Example 3.2). Therefore the claim follows from Theorem 2.11. □

MV-algebras and Riesz MV-algebras are the varieties generated by their standard models $[0,1]$, hence we can safely discuss about the free objects in such varieties. By general results in Universal Algebra, free MV (Riesz) algebras $k$-generated are the subalgebras of $([0,1])^{[0,1]^k}$ generated by the projection maps [7].

More specifically, McNaughton's theorem [37] states that the free $k$-generated MV-algebra, which is the Lindenbaum-Tarski algebra of Łukasiewicz logic, is (up to isomorphism) the algebra of functions from $[0,1]^k$ to $[0,1]$ that are piecewise linear with integer coefficients. A similar result holds for Riesz MV-algebras [13, Theorem 11]. In this case the free object is the algebra of functions which are piecewise linear with *real* coefficients. In what follows we will denote the free $k$-generated MV-algebra by $\mathbf{M}(k)$, while $\mathbf{M}_R(k)$ denotes the free $k$-generated Riesz MV-algebra.

We close this section by proving that no countable MV-algebra – and in particular no free MV-algebra – can be endowed with a CMV-structure.

As a preliminary remark observe that the following is an immediate consequence of the termwise equivalence between CMV-algebras and Riesz MV-algebras and in particular of Corollary 4.4.

**Proposition 4.5.** *Let* $(\mathbf{A}, \mathscr{C})$ *be a CMV-algebra and let* $\Phi : [0,1]_{MV} \to \mathbf{A}$ *the map defined by* $\Phi(\alpha) = cc_\alpha(1,0)$. *Then* $\Phi$ *is an embedding of MV-algebras.*

*Proof.* First, $\Phi(0) = cc_0(1,0) = 0$ by (C1). Let $\alpha \in [0,1]$, $\Phi(1-\alpha) = cc_{1-\alpha}(1,0) = cc_\alpha(0,1) = cc_\alpha(1^*,0^*) = cc_\alpha(1,0)^* = \Phi(\alpha)^*$.

We first remark that, in $[0,1]_{MV}$, $x \oplus y = \min(x+y,1) = \min(x+y, 1-x+x) = x + \min(y, 1-x) = x + (x^* \wedge y)$, whence

$$x \oplus y = x + (x^* \wedge y). \tag{VII}$$

Being the variety of MV-algebras generated by $[0,1]_{MV}$, (VII) holds in every MV-algebra. In order to prove that $\Phi(\alpha \oplus \beta) = \Phi(\alpha) \oplus \Phi(\beta)$ we will show that $\Phi$ commutes on $+$ and $\wedge$. Hence we enter a case distinction:

(i) $\Phi(\alpha+\beta) = cc_{\alpha+\beta}(1,0) = cc_\alpha(1,0) + cc_\beta(1,0) = \Phi(\alpha) + \Phi(\beta)$ by (C5).

(ii) Let $\alpha, \beta \in [0,1]$. Without loss of generality let us assume $\alpha \leq \beta$. Then $\Phi(\alpha \wedge \beta) = \Phi(\alpha) = cc_\alpha(1,0)$. On the other hand $cc_\alpha(1,0) \leq cc_\beta(1,0)$ by Proposition 3.3(v), hence $\Phi(\alpha) \wedge \Phi(\beta) = \Phi(\alpha)$.

Finally, as to prove that $\Phi$ is an embedding, Proposition 3.5 ensures that $\Phi(\alpha) = cc_\alpha(1,0) = 1$ iff $\alpha = 1$. Hence $\Phi^{-1}(1) = \{1\}$. □

**Remark 4.6.** An immediate consequence of Proposition 4.5 states that the MV-reduct $\mathbf{A}$ of a CMV-algebra always contains a copy of the standard MV-algebra $[0,1]_{MV}$. Therefore, no countable MV-algebra – and in particular no free MV-algebras – can be endowed with a CMV-structure. This remark will turn out to be useful in the following Section 5 where we will deal with an algebraic representation of states.

## 5. Expected values as terms in CMV-algebras

In this section we will always deal, unless otherwise specified, with finite dimensional MV-algebras (recall Example 2.2(2)). Hence $\mathbf{A}$ will denote, for a finite set $X$, the MV-algebra $[0,1]^X_{MV}$. Let $k = |X|$, let $[k] = \{1, \ldots, k\}$ and denote, for all $i \in [k]$, $\pi_i : \mathbf{A} \to [0,1]_{MV}$ the $i$th projection map. It is well-known that $\{\pi_i : A \to [0,1] \mid i \in [k]\}$ coincides with the set $\mathcal{H}(\mathbf{A})$ of MV-homomorphisms of $\mathbf{A}$ in the standard algebra $[0,1]_{MV}$. In what follows, for every $k \in \mathbb{N}$, we shall denote $\Lambda(k)$ the set of all probability distributions $p : [k] \to [0,1]$.

Homomorphisms of $\mathbf{A}$ and states of $\mathbf{A}$ are maps from $A$ to $[0,1]$. Hence, the smallest MV-algebra that contains $\mathcal{H}(\mathbf{A})$ is the MV-subalgebra of $[0,1]^A$ generated by $\mathcal{H}(\mathbf{A})$, that is the free MV-algebra $k$-generated $\mathbf{M}(k)$ (see Example 2.2 (3) and [11, 43]).

As we already noticed in Remark 4.6, $\mathbf{M}(k)$ is not closed under convex combinations, but on the other hand every Riesz MV-algebra defines convex combinations as we proved in Theorem 4.3. Let hence $R(\mathbf{M}(k))$ be the *Riesz hull* of $\mathbf{M}(k)$ as defined in [15] i.e., the smallest Riesz MV-algebra containing $\mathbf{M}(k)$. From [32], we know that

$$R(\mathbf{M}(k)) \cong \mathbf{M}_R(k),$$

where the latter denotes the free Riesz MV-algebra $k$-generated. Finally, let $(\mathbf{M}_R(k), \mathscr{C})$ be $\mathcal{C}(\mathbf{M}_R(k))$, where the map $\mathcal{C}$ is defined as in Section 4.

Let us introduce the following notation: for every $\alpha_1, \ldots, \alpha_{k-1} \in [0,1]$ we set $\overline{\alpha} = \alpha_1, \ldots, \alpha_{k-1}$. Since we will always consider strings of length $k-1$, this notation is used without danger of confusion. Further, for every $f_1, \ldots, f_k \in \mathbf{M}_R(k)$, we write

$$cc_{\overline{\alpha}}(f_1, \ldots, f_k) \text{ for } cc_{\alpha_1}(f_1, cc_{\alpha_2}(f_2, \ldots cc_{\alpha_{k-1}}(f_{k-1}, f_k) \ldots)).$$

In other words, $cc_{\overline{\alpha}}(f_1, \ldots, f_k)$ stands for the *nested* convex combination of $f_1, \ldots, f_k$ with parameters $\alpha_1, \ldots, \alpha_{k-1}$. By Theorem 4.3, $cc_{\overline{\alpha}}(f_1, \ldots, f_k)$ is equivalent, in $\mathbf{M}_R(k)$, to the following term:

$$\alpha_1 f_1 + (1 - \alpha_1)\alpha_2 f_2 + \ldots$$

$$+(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{k-2})\alpha_{k-1} f_{k-1} + (1 - \alpha_1) \ldots (1 - \alpha_{k-1}) f_k.$$

Hence,

$$cc_{\overline{\alpha}}(f_1, \ldots, f_k) = \left( \sum_{i=1}^{k-1} \left( \prod_{j=0}^{i-1} (1 - \alpha_j)\alpha_i \right) f_i \right) + \left( \prod_{j=1}^{k-1} (1 - \alpha_j) \right) f_k, \quad \text{(VIII)}$$

where we set $\alpha_0 = 0$.

**Proposition 5.1.** *For every $\alpha_1, \ldots, \alpha_{k-1} \in [0, 1]$, $cc_{\overline{\alpha}}(\pi_1, \ldots, \pi_k)$ is a state of $\mathbf{A}$. If $0 < \alpha_i < 1$ for every $i = 1, \ldots, k-1$, $cc_{\overline{\alpha}}(\pi_1, \ldots, \pi_k)$ is faithful.*

*Proof.* As to prove the first part of the claim, let us proceed by induction on $k$. If $k = 1$, $cc_{\overline{\alpha}}(\pi_1, \ldots, \pi_k) = \pi_1$ is an MV-homomorphism in $[0, 1]_{MV}$, whence a state. In general, we only need to prove that for every every pair of states $s_1, s_2$, and for every $\alpha \in [0, 1]$, the map $\sigma : a \in A \mapsto cc_\alpha(s_1(a), s_2(a))$ is a state of $\mathbf{A}$. The latter claim actually trivially holds since, by Example 3.2 (i), and for every $a \in A$,

$$cc_\alpha(s_1(a), s_2(a)) = \alpha s_1(a) + (1 - \alpha)s_2(a)$$

in $[0, 1]_{PMV}$. Since states are closed under convex combinations, $\sigma$ is a state.

Finally, recalling Equation VIII, if $\alpha_1, \ldots, \alpha_k \in (0, 1)$, $\prod_{j=0}^{i-1}(1-\alpha_j)\alpha_i > 0$ and $\prod_{j=1}^{k-1}(1 - \alpha_j) > 0$. Hence, assuming that $cc_{\overline{\alpha}}(\pi_1, \ldots, \pi_k)(a) = 0$, then

$$\left( \sum_{i=1}^{k-1} \left( \prod_{j=0}^{i-1} (1 - \alpha_j)\alpha_i \right) \pi_i(a) \right) + \left( \prod_{j=1}^{k-1} (1 - \alpha_j) \right) \pi_k(a) = 0$$

and hence $\pi_i(x)$ is necessarily 0 for all $i = 1, \ldots, k$, that is to say, $a = 0$ and $cc_{\overline{\alpha}}(\pi_1, \ldots, \pi_k)$ is faithful. □

Now we are going to show the converse of the previous Proposition 5.1, i.e. that every state of $\mathbf{A}$ can be expressed in the algebraic language of CMV-algebras. It is interesting to point out that a similar universal algebraic approach to states of MV-algebras was developed in [20]. There, states are treated as internal – modal – operators. On the other hand, the result we are going to exhibit provides and algebraic *definition* of states. First of all we need to show a preliminary result.

**Lemma 5.2.** *Let $\lambda_1, \ldots, \lambda_k \in [0,1]$ be such that $\sum_i \lambda_i = 1$. Then, there are $\alpha_1, \ldots, \alpha_{k-1} \in [0,1]$ such that*

*(a) For all $i = 1, \ldots, k-1$, $\lambda_i = \alpha_i \cdot \prod_{j<i}(1-\alpha_j)$,*

*(b) $\lambda_k = \prod_{j<k}(1-\alpha_j)$.*

*Proof.* Let us put $\alpha_0 = 0$ and for each $i \neq 0$ we define $\alpha_i$ in such a way that (a) is satisfied for every $i \in [k-1]$. In particular let us start defining $\alpha_1 = \lambda_1$. As for $i = 2$ we want the equation $\lambda_2 = \alpha_2(1-\alpha_1)$ to be satisfied. But since $\alpha_1 = \lambda_1$, we have that $\lambda_2 = \alpha_2(1-\lambda_1)$ whence it is sufficient to define $\alpha_2 = \lambda_2/(1-\lambda_1)$. As for $i = 3$, in a similar way, we want

$$\lambda_3 = \alpha_3(1-\alpha_2)(1-\alpha_1) = \alpha_3\left(1 - \frac{\lambda_2}{(1-\lambda_1)}\right)(1-\lambda_1)$$

to hold, whence we set $\alpha_3 = \frac{\lambda_3}{\left(1-\frac{\lambda_2}{(1-\lambda_1)}\right)(1-\lambda_1)}$. In general, for every $i \in [k-1]$, condition $(a)$ states that $\lambda_i = \alpha_i \cdot X(\alpha_1, \ldots, \alpha_{i-1})$ where $X(\alpha_1, \ldots, \alpha_{i-1})$ is a term only depending on $\alpha_1, \ldots, \alpha_{i-1}$. Inductively for $j \leq i-1$, the $\alpha_j$ is defined in terms of $\lambda_1, \ldots, \lambda_j$ as, say, a term $Y(\lambda_1, \ldots, \lambda_j)$. Hence

$$X(\alpha_1, \ldots, \alpha_{i-1}) = X(Y(\lambda_1), Y(\lambda_1, \lambda_2), \ldots, Y(\lambda_1, \ldots, \lambda_{i-1})).$$

Thus

$$\alpha_i = \frac{\lambda_i}{X(Y(\lambda_1), Y(\lambda_1, \lambda_2), \ldots, Y(\lambda_1, \ldots, \lambda_{i-1}))} = \frac{\lambda_i}{\sum_{j \geq i} \lambda_j} \qquad \text{(IX)}$$

**Fact 1.** $\sum_{j\geq i} \lambda_j = \prod_{j<i}(1-\alpha_j)$.

*Proof.* (of Fact 1) Let us prove the claim by induction on $i$. If $i = 1$, then $\sum_{j\geq 1} \lambda_j = 1$ and $\prod_{j<1}(1-\alpha_j) = 1 - 0 = 1$. Let us assume the claim holds for $i$ and let us show the case $i+1$.

$$\begin{array}{rcl}
\prod_{j<i+1}(1-\alpha_j) & = & (1-\alpha_i) \cdot \prod_{j<i}(1-\alpha_j) \\
& = & (1-\alpha_i) \cdot \sum_{j\geq i} \lambda_i \\
& = & \left(1 - \frac{\lambda_i}{\sum_{j\geq i}\lambda_j}\right) \cdot \sum_{j\geq i} \lambda_j \\
& = & \sum_{j\geq i} \lambda_j - \lambda_i \\
& = & \sum_{j\geq i+1} \lambda_j,
\end{array}$$

where in the second equality we used the inductive hypothesis while the third one holds because of Equation (IX). □

Turning back to the proof of Lemma 5.2, notice that $\lambda_k = \sum_{j\geq k}\lambda_j$ and hence (a) and (b) hold from Fact 1 and (IX). □

The following proposition shows the converse of Lemma 5.2 above.

**Proposition 5.3.** *Let* $\lambda_1,\dots,\lambda_k \in [0,1]$ *be such that* $\sum_i \lambda_i = 1$. *Let* $\alpha_1,\dots,\alpha_{k-1} \in [0,1]$ *be such that*

*(i) For all* $i = 1,\dots,k-1$, $\lambda_i = \alpha_i \cdot \prod_{j<i}(1-\alpha_j)$,

*(ii)* $\lambda_k = \prod_{j<k}(1-\alpha_j)$.

*Then* $\alpha_i = \frac{\lambda_i}{1-\sum_{j<i}\lambda_j} = \frac{\lambda_i}{\sum_{j\geq i}\lambda_j}$.

*Proof.* The same Fact 1 can be proved within the hypothesis of Proposition 5.3. Indeed,

**Fact 2.** $\sum_{j\geq i}\lambda_j = \prod_{j<i}(1-\alpha_j)$.

*Proof.* (of Fact 2) By (reverse) induction on $i$. Obviously the claim is true for $i = k$ (hypothesis (ii) of Proposition 5.3). Let hence assume that the claim holds for $i+1$. Then

$$\begin{aligned}
\textstyle\sum_{j\geq i}\lambda_i &= \lambda_i + \textstyle\sum_{j\geq i+1}\lambda_j \\
&= \alpha_i \textstyle\prod_{j<i}(1-\alpha_j) + \prod_{j\leq i+1}(1-\alpha_j) \\
&= \alpha_i \textstyle\prod_{j<i}(1-\alpha_j) + (1-\alpha_i)\prod_{j<i}(1-\alpha_j) \\
&= \textstyle\prod_{j<i}(1-\alpha_j).
\end{aligned}$$

Hence, Fact 2 holds. □

Turning back to the proof of our claim, Fact 2 ensures the following

$$\frac{\lambda_i}{\sum_{j\geq i}\lambda_j} = \frac{\alpha_i \prod_{j<i}(1-\alpha_j)}{\prod_{j<i}(1-\alpha_j)} = \alpha_i.$$

Thus our claim is settled. □

The next result provides an algebraic characterization of states of finite dimensional MV-algebras within finitely generated free CMV-algebras.

**Theorem 5.4.** *A map* $s : A \to [0,1]$ *is a state iff there are* $\alpha_1,\dots,\alpha_{k-1} \in [0,1]$ *such that, for every* $a \in A$, $s(a) = cc_{\overline{\alpha}}(\pi_1,\dots,\pi_k)(a)$.

*Proof.* The right-to-left direction has been proved in Proposition 5.1. Hence, let $s$ be a state of $\mathbf{A} = [0,1]^k$. Then, by Theorem 2.7 (see also equation (II)), there exists a unique $p = (\lambda_1, \ldots, \lambda_k) \in \Lambda(k)$ such that, for every $a \in A$, $s(a) = \sum_{i=1}^{k} \pi_i(a)\lambda_i$. Let, for all $i \in [k-1]$, $\alpha_i = \frac{\lambda_i}{\sum_{j \geq i} \lambda_j}$. Hence put $\Phi : A \to [0,1]$ be the function $cc_{\overline{\alpha}}(\pi_1, \ldots, \pi_k)$. Then, for every $a \in A$,

$$\begin{array}{rcl}\Phi(a) & = & cc_{\overline{\alpha}}(\pi_1, \ldots, \pi_k)(a) \\ & = & \left(\sum_{i=1}^{k-1} \left(\prod_{j=0}^{i-1}(1-\alpha_j)\alpha_i\right) a_i\right) + \left(\prod_{j=1}^{k-1}(1-\alpha_j)\right) a_k.\end{array}$$

and hence, by Lemma 5.2, $\Phi(a) = \sum_{i=1}^{k} \lambda_i \pi_i(a) = s(a)$.

□

## 6. A logico-algebraic perspective on the Anscombe-Aumann Representation

We conclude this paper by showing how the framework of CMV-algebras can provide a fresh, logico-algebraic, perspective on the foundations of decision theory under uncertainty, and in particular to the Anscombe-Aumann representation theorem. We insist on this, rather than on the original setting due to Savage, which we nonetheless briefly recall below, for the following twofold reason. Not only does the Anscombe-Aumann set up provide an elegant presentation of the classic justification for the probabilistic quantification of uncertainty, it also constitutes a solid step-stone for the extension of the method to non-probabilistic measures of uncertainty, as comprehensively illustrated in [23][6]. Future work will tell us whether the investigation of the analogues of Theorem 6.2 for non-additive, set-valued measures of uncertainty and more generally, non-expected utility theory, will yield significant improvements in our understanding of norms of reasoning and decision-making under uncertainty. The result presented in this concluding Section certainly provides good grounds for optimism.

### 6.1. Preliminaries

Readers familiar with this material may quickly move to the next subsection. We urge the remaining readers to consult [8, 29] for precise details and historical background.

[6] In addition, the Anscombe-Aumann representation holds unproblematically in *finite* spaces. This is not the case, in general, for the Savage representation. See [25] for a compact presentation of the issue.

Recall that the representation of *consistent preferences* via real-valued utility functions plays a central role in mathematical economics and, more generally, throughout the formal social sciences. The recent history of the field starts with the von Neumann-Morgenstern representation [45] which provides a major improvement on the classical justification of "expected utility" based on repeated independent and identically distributed trials. By combining the von Neumann-Morgenstern setting with the subjectivist foundation of probability, the Savage representation theorem [47] provides an even more general justification for the expected utility criterion which further dispenses with the assumption that a probability distribution is *given* for each decision problem of interest. Rather it is the consistency of the agent's preferences which guarantees the existence of a unique *subjective* probability measure $p$ on the set of all relevant eventualities and a unique (up to positive affine transformations) utility function $u$ over the outcomes. Informally then the Savage representation amounts to establishing the logical equivalence between (i) the maximisation of the expectation functional which arises by composing $p$ and $u$ above and (ii) the axiomatically defined consistency of the agent's preferences over uncertain prospects.

Let $\Sigma$ be a finite set of "states of the world" which we refer to as *Savage-states* and let $\mathcal{C}$ be a set of "consequences", loosely interpreted as what happens to the agent after they take a certain course of action 'in the real world'. All the maps from $\Sigma$ to $\mathcal{C}$ form the set $\mathcal{F}$ of *acts*. So, $f(\sigma) = c$ reads informally as "consequence $c$ is the result of having chosen $f$ when $\sigma$ occurs", e.g. "getting wet"($c$) when choosing "to walk home" ($f$) and "it rains" ($\sigma$).

**Remark 6.1.** Note that there is some unavoidable risk of terminological confusion with "states". Savage-states constitute the elements of what is commonly referred to as the "state-space". This formalises the elementary 'states of the world' which are relevant to modelling a given decision problem under uncertainty. This has nothing to do with the notion of states on MV-algebras captured by Definition 2.5 above.

Acts in $\mathcal{F}$ represent the objects of choice for the decision maker, i.e. those uncertain prospects over which the agent has well-defined *preferences.* So in the above situation the agent may be thought of naturally as having preferred $f$ to "taking a taxi" ($g$). Preferences are modelled by defining a binary relation $\succsim$ on $\mathcal{F}$. Now the Savage representation theorem identifies the conditions on $\succsim$, sometimes referred to as the *consistency axioms*, which are necessary and sufficient to ensure the existence of a unique probability function $p : 2^{\Sigma} \to [0, 1]$ and a (cardinally unique) utility function $u : \mathcal{C} \to$

$[0,1]$ such that

$$f \succsim g \Leftrightarrow E_p(f) \geq E_p(g)$$

where $E_p(f)$ is the *expected utility of* $f$ under $p$, i.e.

$$E_p(f) = \sum_{\sigma\in\Sigma} U(f)(\sigma). \tag{X}$$

Note that the expected utility index $U(\cdot)$ can be written as

$$U(f)(\sigma) = \sum_{x\in supp\, f(\sigma)} u(x)f(\sigma)$$

and represents the objective expected utility of $f(\sigma)$.

In their very influential paper [1], Anscombe and Aumann provide a significant simplification of the Savage representation theorem which takes mathematical advantage of the geometric properties of von Neumann-Morgenstern *lotteries*, which we now introduce. Informally, a lottery consists of a set of 'tickets' each yielding a (sure monetary) prize with a given objective probability. For definiteness, think of a roulette table as the lottery $\lambda = (x_1 : p_1; \dots x_n : p_n)$ in which each $p_i$ stands for the probability of getting prize $x_i$, etc.

The *lottery space* $\Lambda(\Sigma)$, is the set of probability distributions over $\Sigma$, i.e.

$$\Lambda(\Sigma) = \left\{ p : \Sigma \to [0,1] \mid \sum_{\sigma\in\Sigma} p(\sigma) = 1 \right\} \tag{XI}$$

The key feature of the Anscombe-Aumann framework is that Savage-consequences $\mathcal{C}$ are replaced with the lottery space $\Lambda(\Sigma)$. Since this latter is indeed a simplex, the objects of choice (i.e. Savage-acts – the set of all maps from states to lotteries) form a convex set. This guarantees that for all $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$ *mixed acts* are always defined as convex combinations of acts $\alpha f + (1-\alpha)g$, such that for all $\sigma \in \Sigma$

$$(\alpha f + (1-\alpha)g)(\sigma) = \alpha f(\sigma) + (1-\alpha)g(\sigma).$$

This creates a natural bridge between the Anscombe-Aumann setting and the framework of CMV-algebras. Before crossing it we need to present the Anscombe-Aumann axioms and state their representation result.[7]

An agent's preference relation over acts $\succsim$ is *consistent* if it satisfies the following conditions:

[7] Readers with an interest to the justification of the axioms and the decision-theoretic interpretation of the result are referred to the original [1] and to the presentations of [8] and [22] for more background.

**A-A.1** WEAK ORDERING

$$\succsim \subseteq \mathcal{F}^2 \text{ is total and transitive}$$

**A-A.2** CONTINUITY (ARCHIMEDEAN) For $f, g, h \in \mathcal{F}$ such that $f \succ g \succ h$, $\exists \alpha, \beta \in (0,1)$ such that

$$f \succ g \Rightarrow \alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h.$$

**A-A.3** INDEPENDENCE For $f, g, h \in \mathcal{F}$, and $\alpha \in (0,1)$,

$$f \succ g \Rightarrow \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h.$$

**A-A.4** MONOTONICITY

$$\text{If } f(\sigma) \succsim g(\sigma) \forall \sigma \in \Sigma, \text{ then } f \succsim g,$$

where we consider the order on $\Lambda(\Sigma)$ induced by the order on $\mathcal{F}$, considering any consequence as a constant act.

**Theorem 6.2** ([1])**.** *The following are equivalent:*

1. $\succsim \subseteq \mathcal{F}^2$ *is consistent*
2. *There exists a cardinally unique* $u : \Lambda(\Sigma) \to [0,1]$ *and a unique* $p : \Sigma \to [0,1]$ *s.t.*
$$f \succsim g \Leftrightarrow E_p(f) \geq E_p(g).$$

## 6.2. The Anscombe-Aumann representation in CMV-algebras

By a *state* of a CMV-algebra $(\mathbf{A}, \mathscr{C})$ we mean a state of its MV-reduct $\mathbf{A}$. It is a direct consequence of Theorem 4.3 that, if $s$ is a state of a CMV-algebra $(\mathbf{A}, \mathscr{C})$, then

$$s(cc_\alpha(x,y)) = cc_\alpha(s(x), s(y)), \text{ for } x, y \in A \text{ and } \alpha \in [0,1]. \tag{XII}$$

Let $\Lambda(k) = \Lambda(\Sigma)$ be as in (XI) above, where $k$ is the cardinality of $\Sigma$.

**Lemma 6.3.** *For every* $k \in \mathbb{N}$, $[0,1]^k$ *is the term CMV-algebra generated by* $\Lambda(k)$.

*Proof.* In order to prove that $[0,1]^k$ coincides with $\langle \Lambda(k) \rangle_{CMV}$, notice that every $b \in 2^k$ (the Boolean skeleton of $[0,1]^k$) is obtainable as an MV-combination of the vertices $e_1, e_2, \ldots, e_k$ of $\Lambda(k)$. Indeed, if $b = (b_1, \ldots, b_k)$ with at least a $b_i \neq 0$, then $b = \bigoplus_{b_i \neq 0} e_i$, where $e_i$ is the vertex of $\Lambda(k)$ made of all 0's, and whose $i$th component is 1. In particular $1 = e_1 \oplus e_2 \oplus \ldots, \oplus e_k$ and $0 = 1^*$.

Now, since $[0,1]^k = \mathrm{co}(2^k)$, for every $x \in [0,1]^k$, there are $\lambda_1, \ldots, \lambda_{2^k} \in [0,1]$ that sum up to 1 and such that $x = \sum_{i=1}^{2^k} \lambda_i \cdot e_i$, where $e_i$ denotes the generic element of $2^k$. Hence, by Lemma 5.2, there are $\alpha_1, \ldots, \alpha_{2^k-1} \in [0,1]$ such that $x = cc_{\overline{\alpha}}(e_1, \ldots, e_{2^k})$, whence the claim is settled.

□

**Corollary 6.4.** *Every map $u : \Lambda(k) \to [0,1]_{CMV}$ extends to a homomorphism of CMV-algebras $l : [0,1]^k \to [0,1]_{CMV}$.*

*Proof.* The claim directly follows from Lemma 6.3 and [7, Theorem 10.8] stating that term algebras have the universal mapping property.

□

Two maps $u$ and $u'$ from $\Lambda(k)$ in $[0,1]$ are *affinely dependent* if $u = \alpha u' + \beta$ for some $\alpha, \beta \in [0,1]$ with $\alpha > 0$. We say that two homomorphisms $l, l' : [0,1]^k \to [0,1]_{CMV}$ are *affinely generated* if their restrictions to $\Lambda(k)$ are affinely dependent. Given a class $H$ of homomorphisms from $[0,1]^k$ to $[0,1]$, we say that $l \in H$ is *cardinally unique* if for any $l' \in H$, $l$ and $l'$ are affinely generated.

For every $k \in \mathbb{N}$, every $f \in \mathcal{F}$ and every homomorphism $l : [0,1]^k \to [0,1]_{CMV}$, let $l_f : [k] \to [0,1]$ be defined by composition: for every $i \in [k]$,

$$l_f(i) = l(f(i)).$$

Notice that if $u : \Lambda(k) \to [0,1]$ is the restriction of $l$ to $\Lambda(k)$, then $u$ is a utility function and $l_f = U(f)$.

We are finally in a position to state a logico-algebraic formulation of the Anscombe-Aumann Representation Theorem.

**Theorem 6.5.** *Let $\succeq$ be a binary relation on $\mathcal{F} = \{f \mid f : [k] \to \Lambda(k)\}$. Then the following are equivalent:*

*(1) $\succeq$ satisfies axioms (A-A.1)-(A-A.4) above,*

*(2) there exist a cardinally unique homomorphism $l : [0,1]^k \to [0,1]_{CMV}$ and a unique state $s : [0,1]^k \to [0,1]$ such that for any $f, g \in \mathcal{F}$*

$$f \succeq g \Leftrightarrow s(l_f) \geq s(l_g).$$

*(3) there exist a cardinally unique homomorphism $l : [0,1]^k \to [0,1]_{CMV}$ and unique $\alpha_1, \ldots, \alpha_{k-1} \in [0,1]$ such that*

$$f \succeq g \Leftrightarrow cc_{\overline{\alpha}}(\pi_1, \ldots, \pi_k)(l_f) \geq cc_{\overline{\alpha}}(\pi_1, \ldots, \pi_k)(l_g).$$

*Proof.* The equivalence between (2) and (3) is an immediate consequence of Theorem 5.4. Let hence prove (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2). From the Anscombe-Aumann theorem, there exists a unique utility function $u : \Lambda(k) \to [0,1]$ and a unique probability measure $p : 2^k \to [0,1]$ such that $f \succeq g$ iff $E_p(f) \geq E_p(g)$.

By Corollary 6.4, $u$ extends to an homomorphism $l : [0,1]^k \to [0,1]_{CMV}$.

Moreover, as we observed above, $l_f = U(f)$. Let $s$ be the state of $[0,1]^k$ obtained from $p$ through the representation of Theorem 2.7. Then for every $f \in \mathcal{F}$,

$$s(l_f) = \sum_{i=1}^{k} p(i) \cdot l_f(i) = \sum_{i=1}^{k} p(i) \cdot U(f)(i) = E_p(f). \tag{XIII}$$

Hence this direction is easily settled.

We now exhibit two ways to prove (1) $\Leftarrow$ (2). While the first one will use again the Anscombe-Aumann theorem, the second proof will show that axioms (A-A.1)–(A-A.4) hold by direct algebraic computations in CMV-algebras.

(1) $\Leftarrow$ (2) (first proof). Let $l$ and $s$ be as in the hypothesis and let $u : \Lambda(k) \to [0,1]$ and $p$ be, respectively, the unique utility function on $\Lambda(k)$ obtained by restriction of $l$, and the unique probability distribution on $[k]$ whose existence is ensured by Theorem 2.7. Then, by a similar argument used to prove the chain of equalities of (XIII), our hypothesis implies that $f \succeq g$ iff $E_p(f) \geq E_p(g)$. Hence, by the Anscombe-Aumann theorem (A-A.1)–(A-A.4) are satisfied.

(1) $\Leftarrow$ (2) (second proof). Let $s$, $l$ be as in the hypothesis. We need to prove that $\succeq$ satisfies (A-A.1)–(A-A.4). Since the range of $s$ is a total and transitive order, Axiom (A-A.1) is easily proved. In order to prove (A-A.2), let $f, g, h \in \mathcal{F}$ be such that $f \succ g \succ h$. Then, by hypothesis, $s(l_f) > s(l_g) > s(l_h)$. Notice that the claim to be proved can be formulated in the language of CMV-algebras and states as follows:

$$s(l_{cc_\alpha(f,h)}) > s(l_g) > s(l_{cc_\beta(f,h)}),$$

where the operators $cc$'s are defined on the CMV-algebra $([0,1]^k)^k$. By the very definition of $l_t$ for $t \in ([0,1]^k)^k$, since $cc_\alpha(t,t') \in ([0,1]^k)^k$ for every $\alpha \in [0,1]$ and every $t, t' \in ([0,1]^k)^k$, equation (XII) and the fact that $l : [0,1]^k \to [0,1]$ is a state of the CMV-algebra $[0,1]^k$ imply

$$l_{cc_\alpha(f,h)} = l \circ cc_\alpha(f,h) = cc_\alpha(l \circ f, l \circ h) = cc_\alpha(l_f, l_h).$$

Hence, from (XII) it follows that $s(l_{cc_\alpha(f,h)}) = s(cc_\alpha(l_f, l_h)) = \alpha s(l_f) + (1 - \alpha)s(l_h)$. Now, let $\alpha > \frac{s(l_g)-s(l_h)}{s(l_f)-s(l_h)}$. Then

$$\alpha(s(l_g) - s(l_h)) > s(l_g) - s(l_h).$$

Re-arranging the terms, we get

$$\alpha s(l_f) + (1 - \alpha)s(l_h) > s(l_g).$$

Moreover, for every $\beta < \frac{s(l_g)-s(l_h)}{s(l_f)-s(l_h)}$, the desired inequality follows by a similar argument.

Let now $f, g, h \in \mathcal{F}$. In order to prove (A-A.3), let $f \succ g$, whence, by hypothesis, $s(l_f) > s(l_g)$. Then for every $\alpha \in (0, 1)$,

$$s(l_{cc_\alpha(f,h)}) = \alpha s(l_f) + (1 - \alpha)s(l_h) > \alpha s(l_g) + (1 - \alpha)s(l_h) = s(l_{cc_\alpha(g,h)}).$$

Hence, $cc_\alpha(f, h) \succ cc_\alpha(g, h)$.

Finally, (A-A.4) follows from the monotonicity of states and homomorphisms of MV-algebras. □

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## Appendix

### Proof of Proposition 3.3

(i) It is a direct consequence of (C1) and (C2). Indeed, $cc_1(x, y) = cc_0(y, x) = x$.

(ii) By (C4), taking $\mu = \frac{\alpha(1-\beta)}{1-\alpha\beta}$, we have $cc_\alpha(cc_\beta(x, y), y) = cc_{\alpha\beta}(x, cc_\mu(y, y))$. Hence, since by (C3) $cc_\mu(y, y) = y$, the claim follows.

(iii) From (C2), $cc_\alpha(cc_\beta(x, z), cc_\gamma(y, z)) = cc_\alpha(cc_{1-\beta}(z, x), cc_\gamma(y, z))$. Thus, letting $r = \frac{\alpha\beta}{1-\alpha+\alpha\beta}$,

(C4) implies that

$$cc_\alpha(cc_{1-\beta}(z, x), cc_\gamma(y, z)) = cc_{\alpha(1-\beta)}(z, cc_r(x, cc_\gamma(y, z))),$$

and

$$cc_{\alpha(1-\beta)}(z, cc_r(x, cc_\gamma(y, z))) = cc_{\alpha(1-\beta)}(z, cc_{1-r}(cc_{1-\gamma}(z, y), x))$$

from (C2). Let $s = \frac{(1-r)\gamma}{1-(1-r)(1-\gamma)} = \frac{(1-\alpha)\gamma}{(1-\alpha)\gamma+\alpha\beta}$. Then, a further instantiation of (C4) gives

$$cc_{\alpha(1-\beta)}(z, cc_{1-r}(cc_{1-\gamma}(z,y),x)) = cc_{\alpha(1-\beta)}(z, cc_{(1-r)(1-\gamma)}(z, cc_s(y,x))).$$

Thus, we finally have

$$cc_{\alpha(1-\beta)}(z, cc_{(1-r)(1-\gamma)}(z, cc_s(y,x))) = \\ cc_{1-\alpha(1-\beta)}(cc_{1-(1-r)(1-\gamma)}(cc_s(y,x),z),z)$$

and by the above proved (ii)

$$cc_{1-\alpha(1-\beta)}(cc_{1-(1-r)(1-\gamma)}(cc_s(y,x),z),z) = \\ cc_{[1-\alpha(1-\beta)][1-(1-r)(1-\gamma)]}(cc_{1-s}(x,y),z).$$

Hence the claim follows by letting $\mu = [1-\alpha(1-\beta)][1-(1-r)(1-\gamma)] = \alpha\beta + (1-\alpha)\gamma$ and $\nu = 1 - s = \frac{\alpha\beta}{\mu}$.

(iv) From (iii) we immediately obtain,

$$cc_\alpha(cc_\beta(a,b), cc_\beta(x,y)) = cc_{\alpha\beta}(a, cc_{(1-\mu)\beta}(x, cc_\nu(y,b))), \qquad \text{(XIV)}$$

with $\mu = \frac{\alpha(1-\beta)}{1-\alpha\beta}$, $1-\mu = \frac{1-\alpha}{1-\alpha\beta}$, $\nu = \frac{(1-\mu)(1-\beta)}{1-(1-\mu)\beta} = \frac{1-\alpha-\beta+\alpha\beta}{1-\beta} = 1-\alpha$.

Therefore, the right hand side of the above equation (XIV) equals

$$cc_{\alpha\beta}(a, cc_{\frac{(1-\alpha)\beta}{1-\alpha\beta}}(x, cc_\alpha(b,y))).$$

In turns, by (C4), the latter also equals

$$cc_\beta(cc_\alpha(a,x), cc_\alpha(b,y)).$$

Hence $cc_\alpha(cc_\beta(a,b), cc_\beta(x,y)) = cc_\beta(cc_\alpha(a,x), cc_\alpha(b,y))$.

(v) Let $\alpha \le \beta$ and let $\gamma \in [0,1]$ be such that $\alpha + \gamma = \beta \le 1$. Then by (C5), $cc_\beta(x,0) = cc_{\alpha+\gamma}(x,0) = cc_\alpha(x,0) \oplus cc_\gamma(x,0) \ge cc_\alpha(x,0)$.

**Proof of Proposition 3.4**

(i) If $x \le x'$, then $x \odot (x')^* = 0$ and $x + (x')^*$ is defined. Moreover $y \odot y^* = 0$, whence $y + y^*$ is defined. Therefore, by (C6), $cc_\alpha(x,y) + cc_\alpha((x')^*, y^*)$ is defined. By (C7), this equals to $cc_\alpha(x,y) + cc_\alpha(x',y)^*$. Hence, by definition of $+$, $cc_\alpha(x,y) \le cc_\alpha(x',y)$.

(ii) It follows from (i) and (C2).

(iii) From (i) and (C3) $cc_\alpha(x,0) \le cc_\alpha(x,x) = x$ and analogously from (ii) and (C3), $cc_\alpha(0,y) \le cc_\alpha(y,y) = y$.

(iv) From (ii) and (iii), $cc_\alpha(x,y) \geq cc_\alpha(x \wedge y, x \wedge y) = x \wedge y$ and for the same reason $cc_\alpha(x,y) \leq x \vee y$. Moreover, (v) holds since, in every MV-algebra $\mathbf{A}$, $x \odot y \leq x \wedge y$ and $x \vee y \leq x \oplus y$.

(vi) and (vii) easily follow from (ii) and (iii).

(viii) If $x \odot z = y \odot z = 0$, from (iv), the motonicity of $\odot$ and the distributivity of $\odot$ of $\vee$ [11, Proposition 1.1.6(i)], $cc_\alpha(x,y) \odot z \leq (x \vee y) \odot z = (x \odot z) \vee (y \odot z) = 0$. Thus $x + z$, $y + z$ and $cc_\alpha(x,y) + z$ are defined and, from (C6), $cc_\alpha(x,y) \oplus z = cc_\alpha(x,y) + z = cc_\alpha(x,y) + cc_\alpha(z,z) = cc_\alpha(x+z, y+z) = cc_\alpha(x \oplus z, y \oplus z)$.

(ix) Since $x \odot 0 = 0 \odot y = 0$, by (C6), $cc_\alpha(x,y) = cc_\alpha(x+0, 0+y) = cc_\alpha(x,0) + cc_\alpha(0,y)$. By (C7), $cc_\alpha(x^*,0) = cc_\alpha(x,1)^*$. Hence,

$$cc_\alpha(x^*,0) = cc_\alpha(x,1)^* = (cc_\alpha(x,0) + cc_\alpha(0,1))^* = cc_\alpha(x,0)^* \odot cc_\alpha(1,0).$$

(x) We first recall that, by [18], $x \vee y = (x \odot y^*) + y$. Hence, by (vi)

$$(cc_\alpha(x,0) \odot cc_\alpha(y,0)^*) + cc_\alpha(y,0) = cc_\alpha(x,0) \vee cc_\alpha(y,0) \leq cc_\alpha(x \vee y, 0) = cc_\alpha(x \odot y^* + y, 0) = cc_\alpha(x \odot y^*, 0) + cc_\alpha(y,0).$$

The conclusion follows now by the cancellation property of the partial sum $+$.

(xi) The Chang distance function $d(x,y)$ (equation (I)) can be equivalently defined by $d(a,b) = (a^* \odot b) \vee (a \odot b^*)$ (cf. [13, Proposition 2.6.1 (d1)]). Thus,

$$d(cc_\alpha(x,0), cc_\alpha(y,0)) = (cc_\alpha(x,0)^* \odot cc_\alpha(y,0)) \vee (cc_\alpha(x,0) \odot cc_\alpha(y,0)^*)$$

By (x) and the commutativity of $\odot$,

$$(cc_\alpha(x,0)^* \odot cc_\alpha(y,0)) \vee (cc_\alpha(x,0) \odot cc_\alpha(y,0)^*) \leq cc_\alpha(x \odot y^*, 0) \vee cc_\alpha(x^* \odot y, 0)$$

Hence, by (vi)

$$cc_\alpha(x \odot y^*, 0) \vee cc_\alpha(x^* \odot y, 0) \leq cc_\alpha((x \odot y^*) \vee (x^* \odot y), 0) = cc_\alpha(d(x,y), 0).$$

The second part of the claim follows from the previous and (C2).

**Proof of Proposition 3.5**

(i) Let $x > 0$ and $\alpha > 0$ and assume, by way of contradiction, that $cc_\alpha(x,0) = 0$. Then, since $cc_\alpha(x,0) + cc_{1-\alpha}(x,0) = cc_1(x,0) = x$, necessarily $cc_{1-\alpha}(x,0) =$

$x > 0$. Now, if $\alpha \geq 1-\alpha$, then we immediately reach a contradiction because by Proposition 3.3 (v), $x = cc_{1-\alpha}(x,0) \leq cc_\alpha(x,0) = 0$ while $x > 0$ by hypothesis.

Let hence assume that $\alpha < 1-\alpha$. Then, $\alpha + (1-2\alpha) = 1-\alpha < 1$ (since $\alpha > 0$). Thus, $x = cc_{1-\alpha}(x,0) = cc_\alpha(x,0) + cc_{1-2\alpha}(x,0)$ which again implies $x = cc_{1-2\alpha}(x,0)$ since $cc_\alpha(x,0) = 0$. Now, as above, if $\alpha \geq 1-2\alpha$ we reach a contradiction. Conversely, if $\alpha < 1-2\alpha$, we proceed as above getting that $x = cc_{1-3\alpha}(x,0)$. In general, if we find a $l \in \mathbb{N}$ such that $\alpha \geq 1-l\alpha$, then

$$0 = cc_\alpha(x,0) \geq cc_{1-l\alpha}(x,0) = x > 0$$

and we reach a contradiction.

Now we prove that such an $l$ exists. Indeed, since $\alpha > 0$, let $m$ be such that $\alpha \geq 1/m = 1-(m-1)(1/m) > 1-(m-1)\alpha$. Let $l = (m-1)$ and from the previous argument we reach a contradiction.

Thus, the claim is settled.

(ii) If $x < x'$, then there exists a $k > 0$ such that $x \oplus k = x + k = x'$ (indeed $k = x' \ominus x$). Thus, from Axiom (C6) it follows that $cc_\alpha(x',y) = cc_\alpha(x+k,y) = cc_\alpha(x,y) + cc_\alpha(k,0) > cc_\alpha(x,y)$ as $cc_\alpha(k,0) > 0$ by (i).

(iii) Let $x, y$ be incomparable. Then $x \wedge y < x, y < x \vee y$. If $\alpha = 0$ the claim trivially follows since $x \wedge y < y < x \vee y$. If $\alpha > 0$, by (ii), $cc_\alpha(x,y) < cc_\alpha(x \vee y, y) \leq cc_\alpha(x \vee y, x \vee y) = x \vee y$. In a complete analogous way we can show that $x \wedge y < cc_\alpha(x,y)$.

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