An approach to stochastic processes via non-classical logic

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Abstract

Within the infinitary variety of σ-complete Riesz MV-algebras $\mathbf{RMV}_\sigma$, we introduce the algebraic analogue of a random variable as a homomorphism defined on the free algebra in $\mathbf{RMV}_\sigma$. After a preliminary study of the proposed notion, we use it to define stochastic processes in the framework of non-classical logic (Łukasiewicz logic, more precisely) and we define stochastic independence.

Keywords: Stochastic processes, non-classical logic, Łukasiewicz logic, MV-algebras, vector lattice, Riesz space, Borel functions, martingale.

Introduction

In 1965 Lofti Zadeh introduced the notion of a fuzzy set, thus laying the ground for the birth of fuzzy logics as we know them. It was soon after their definition that it was raised the question of the relation between fuzzy sets and probabilities. Zadeh itself argued in many papers that, in fact, probability theory and fuzzy logic are complementary rather than competitive, putting the accent on the fact that there are limitations in basing probability on classical logic, while a more effective logical framework can be obtained using fuzzy logics, see e.g. [28, 29].

As research on fuzzy logic evolved, it became more and more clear that MV-algebras, the algebraic semantics of Łukasiewicz logic, could provide the most effective framework to deal with probabilistic reasoning. Many approaches have been pursued over the years, but for the content of this paper the most important steps towards a logical point of view of probability are [20, 17, 23, 21]. Altogether these results achieve two main goals: they successfully define probability measures on MV-algebras, and they do so *subjectively*, in the sense of

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the subjective point of view on probability introduced by Bruno de Finetti. As a consequence, MV-algebras come to be an obvious choice when one wants to investigate probability theory taking into account the uncertainty and fuzziness of natural language. The algebraic counterpart of probability measures are called states, and they were introduced as averaging processes for formulas in Łukasiewicz logic. Moreover, following ideas related to quantum systems, there were subsequently defined various notions of *observables*, meant to model random variables on MV-algebras, see e.g. [9, 10, 16, 24, 25, 27] and the references within. We argue here that these notions of observables, although interesting in themselves, do not play the role of random variables in logical terms. Moreover, if one looks at the recent literature on MV-algebras (here we limit ourselves to mention [22] and the references within), one can see that while there is consensus and interest towards states on MV-algebras, there is not much recent literature (nor consensus) on observables on MV-algebras, especially if one wants *codify* random variables and other key notions of probability in a logical system.

Thus, in this paper we build on this idea and the results of [6] to give a completely logical point of view on random variables and stochastic processes. In particular, we work in a richer language than the one of MV-algebras: other that sum and complement, we allow multiplication by scalars in $[0, 1]$ and countable suprema and infima (thought as operations of countable arity). In fact, we work within σ-complete Riesz MV-algebras, which were proved in [6] to be the infinitary variety, $\mathbf{RMV}_\sigma$, of models of a logical system, $\mathcal{IR}$Ł, that is a conservative extension of Łukasiewicz logic and it is standard complete. Moreover, $\mathbf{RMV}_\sigma$, as a category, is equivalent to a subcategory of unital and commutative C^*-algebras [4], which makes this approach interesting from a physics-based point of view. Indeed states and observable were originally defined having in mind notions from physics and in [22, Section 1.6] one can find a translation of the basic elements of de Finetti's coherence criterion in the commutative C^*-algebraic formulation of classical physical systems.

Our definition of observables is given in Definition 2.1: for us, an *observable* (or *non-classical random variable*) is a homomorphism from the free object in $\mathbf{RMV}_\sigma$ to any σ-complete Riesz MV-algebra. Thus, the approach we pursue here is different in spirit from the existing literature. The key idea is that, in order to discuss probability within a logical system it is necessary first to agree on the definition of an *event*, and we claim that an event should be a formula in the logical system $\mathcal{IR}$Ł. The reason for this choice is the fact that logical formulas should serve as a way of coding an event that is being described to us by sentences of natural language. Starting here, it is only natural to define the algebraic counterpart of a random variable as done in Definition 2.1: the behaviour of a generic event φ is completely understood once we describe it on the propositional variables that appear in φ.

There is, finally, yet another reason for arguing that random variables in logic can be successfully describes in this way: with an eye towards a metamatemathics of probability theory, our notion of observable is internal within the variety $\mathbf{RMV}_\sigma$, as it corresponds to evaluations of formulas. It is worth mentioning that in [11] the authors are able to characterize states (thus, probabilities) of

finite dimensional Riesz MV-algebras by convex combinations, without the need for further symbols in an eventual language. From this point of view, our definition obtains a similar result by keeping the notion of random variable fully described inside the variety $\mathbf{RMV}_\sigma$.

Whence, in this work we investigate the effectiveness and applicability of our definition of random variable in $\mathbf{RMV}_\sigma$. After some needed preliminary notions, observables are introduced in Section 2, where the crucial result is Theorem 2.3. This theorem is a sort of representation theorem, since it gives a way of describing the uncertainty of an event in two distinct layers: the first layer is a purely probabilistic one, while the second is the fuzzy component (or uncertainty) of the event itself. At the end of Section 2, an analysis of the relation of our notion with the already existing ones is carried. Section 3 is devoted at obtaining an integral representation that is in the spirit of the spectral resolution of a Hilbert space, while Section 4 builds on Theorem 2.3 in order to generalize the classical construction of the conditional expectation operator in a measure space. Finally, in Section 5 non-classical stochastic processes are defined with an eye to stochastic independence.

1. Preliminaries

The main characters of this paper are *σ-complete Riesz MV-algebras*. These are algebras that model a conservative extension of Łukasiewicz logic and can be thought as unit intervals of Dedekind σ-complete vector lattices with strong unit [5, 7]. We briefly sketch some needed notions, but we urge the interested reader to consult [6, 7, 19, 22] for a more detailed account of these topics.

A *Riesz MV-algebra* is an algebra $(A, \oplus, \neg, 0, 1, \{\alpha\}_{\alpha\in[0,1]})$, where $\oplus$ is a binary operation, $\neg$ is an involution, 0 and 1 are respectively a bottom and a top element, and the unary operations $\{\alpha\}_{\alpha\in[0,1]}$ model a scalar multiplication. The standard example of such an algebra is the real interval $[0, 1]$, where $x \oplus y = \min(x + y, 1)$, $\neg x = 1 - x$ and αx is the product of real numbers. This example is standard in very precise sense: Riesz MV-algebras form a variety and $[0, 1]$ is a generator for it. From a different point of view, if (V, u) is a Riesz space (vector lattice) with a distinguished strong order unit u, the interval $[0, u]_V = \{x \in V \mid 0 \leq x \leq u\}$ is a Riesz MV-algebra when endowed with the following operations: $x \oplus y = (x +_V y) \wedge u$, $\neg x = u -_V x$, αx the same as in V. The map that takes (V, u) and sends it into $[0, u]_V$ is actually a functor, denoted by Γ, that gives a more general categorical equivalence between Riesz MV-algebras and Riesz spaces with strong unit [7].

A Riesz MV-algebra is σ-complete if it is closed under countable suprema and infima. It was proved in [6] that σ-complete Riesz MV-algebras form an *infinitary* variety in the sense of [26], and the algebra $[0, 1]$ is again a generator for the infinitary variety, subsequently denoted by $\mathbf{RMV}_\sigma$. Moreover, in [6], it was proved that the free n-generated algebra in $\mathbf{RMV}_\sigma$ coincides with the algebra of all Borel-measurable functions $a\colon [0, 1]^n \to [0, 1]$, subsequently denoted by $\mathrm{Borel}([0, 1]^n)$. This result is generalized to the κ-generated case in [4], where the authors prove that the ω-generated free algebra in $\mathbf{RMV}_\sigma$ is

again Borel($[0,1]^\omega$), while for a cardinal $\kappa > \omega$, the free object is the algebra of *Baire*-measurable functions. The algebra Borel($[0,1]^\kappa$), with $\kappa \leq \omega$, is also isomorphic to the Lindenbaum-Tarski algebra of the infinitary logic $\mathcal{IR}$Ł, that extends Łukasiewicz logic by enough connectives to model the scalar multiplication by each real number, and one connective of countable arity to model a countable disjunction.

For any non-empty set X, a *Riesz tribe* is a Riesz MV-subalgebra of $[0,1]^X$ closed under pointwise countable suprema. For the purpose of this paper, we stress the fact that Borel($[0,1]^\kappa$) is a Riesz tribe for any κ, and any σ-complete Riesz MV-algebra is a σ-homomorphic image of a Riesz tribe. This is the so-called Loomis–Sikorski Theorem, see [6, Theorem 5.2], and this fact will be made more precise later on, see Proposition 1.1. If $\mathcal{T} \subseteq [0,1]^X$ is a Riesz tribe, the set $\mathcal{S}(\mathcal{T}) = \{A \subseteq X \mid \chi_A \in \mathcal{T}\}$ is a natural σ-algebra of subsets of X, and it is the smallest σ-algebra of X that makes all functions in $\mathcal{T}$ measurable. Notice that we will always consider the codomain $[0,1]$ of each function in $\mathcal{T}$ endowed with its standard Borel σ-algebra $\mathcal{B}([0,1])$, that is, the σ-algebra generated by the open subsets in the euclidean space $[0,1]$.

We say that a topological space $X \neq \emptyset$ is *basically disconnected* provided the closure of every open F_σ subset (i.e. a countable union of closed subsets) of X is open. By a well-known result of Nakano, if X is a compact Hausdorff space, the Riesz space $C(X)$ (of real-valued continuous functions on X) is Dedekind σ-complete if, and only if, X is basically disconnected, see e.g. [19, Theorem 43.9]. Moreover, by the fundamental Stone-Krein-Kakutani-Yosida Theorem [19, Theorem 45.4] any Dedekind σ-complete unital Riesz space (R, u) is isomorphic (as a Dedekind σ-complete Riesz space) to the unital Riesz space $(C(X), 1_X)$, where X is a basically disconnected compact Hausdorff space and 1_X is the constant function on X identically 1. In addition, X can be chosen to be the set of all maximal ideals of R endowed with the hull–kernel topology. Moreover, the well-known Banach-Stone theorem for Riesz spaces [3, Corollary 12.4], entails that for X and Y non-empty compact Hausdorff topological spaces, $C(X)$ and $C(Y)$ are isomorphic if and only if X and Y are homeomorphic. We stress the fact that countable suprema on continuous functions need not to be pointwise.

Summarizing these facts, we obtain the proposition below.

Proposition 1.1. *Let $X, Y \neq \emptyset$ be compact Hausdorff topological spaces. The following statements hold:*

(i) *The Riesz MV-algebras $\Gamma(C(X), 1_X)$ and $\Gamma(C(Y), 1_Y)$ are isomorphic if, and only if, X and Y are homeomorphic.*

(ii) *X is basically disconnected if, and only if, $\Gamma(C(X), 1_X)$ is a σ-complete Riesz MV-algebra.*

(iii) *$R \in \mathbf{RMV}_\sigma$ if, and only if, there exists a basically disconnected, compact Hausdorff space X such that $R \simeq \Gamma(C(X), 1_X)$. X is the space $Max(R)$ of maximal ideals of R.*

(iv) *For any $R \in \mathbf{RMV}_\sigma$, there exists a Riesz tribe $\mathcal{T} \subseteq [0,1]^X$ such that R is a homomorphic image of $\mathcal{T}$. The triple $(\mathcal{T}, X, \eta)$, with $\mathcal{T} \subseteq [0,1]^X$, $X = Max(R)$ and $\eta : \mathcal{T} \twoheadrightarrow R$ will be called a* tribe representation *of R.*

These results were mentioned in [6, Section 4], but we include them with the aim of giving a better understanding of σ-complete Riesz MV-algebras. Indeed, for the purpose of this paper, it is enough to think of σ-complete Riesz MV-algebras either as algebras of $[0,1]$-valued continuous functions over a suitable space X or as Riesz tribes. We also mention that the tribe representation that will be always used subsequently is the one given in [22, Theorem 11.7].

On MV-algebras and Riesz MV-algebras, a theory of subjective probability has been developed using the notion of a *state*, introduced by D. Mundici in [20] with the idea of obtaining an averaging process for formulas in Łukasiewicz logic. The definition is easily generalized to Riesz MV-algebras, and a *state* of a Riesz MV-algebra A is a map $s : A \to [0,1]$ satisfying the following conditions:

(1) $s(1) = 1$,

(2) for all $x, y \in A$ such that $x \odot y = 0$, $s(x \oplus y) = s(x) + s(y)$.

Thus, states on Riesz MV-algebras do not require additional conditions for the scalar product, see also [7]. We recall that $x \odot y$ is defined as $\neg(\neg x \oplus \neg y)$. A state is σ-additive if it preserves countable suprema of non-decreasing sequences. We also recall that any non-trivial MV-algebra A (i.e. $A \neq \{0\}$) carries at least one state. The requirement $x \odot y = 0$ is the many-valued analogue of disjointness of a pair of elements in a Boolean algebra, and states indeed provide a generalization of finitely additive probabilities to the realm of MV-algebras. More precisely, states on A (endowed with the weak-convergence topology) are homeomorphic to Borel-regular probability measures on $Max(A)$, see [12, Theorem 4.0.1].

States are examples of *additive* functions: given two MV-algebras A, B we call a function $\omega : A \to B$ additive if $\omega(a_1 + a_2) = \omega(a_1) + \omega(a_2)$. The operation $+$ is a partial operation defined when $a_1 \odot a_2 = 0$ and in that case $a_1 + a_2 := a_1 \oplus a_2$. More generally, a function $\omega : A_1 \times \ldots \times A_n \to B$ is n-additive if it is additive in each argument.

Finally, we recall that states enjoy a crucial functional representation [2, 17, 23], that we shall make explicit in the next theorem only in the case of tribes, and it represents a many-valued version of the Riesz representation theorem for linear functionals. In what follows, for any topological space X, $\mathcal{B}(X)$ will always denote the Borel σ-algebra of X generated by its open sets.

Theorem 1.2. *For every Riesz tribe $\mathcal{T} \subseteq [0,1]^X$, for every σ-additive state s of $\mathcal{T}$, for every $f \in \mathcal{T}$,*

$$s(f) = \int_X f \, \mathrm{d}\mu_s.$$

The measure $\mu_s : \mathcal{S}(\mathcal{T}) \to [0,1]$ is given by $\mu(A) = s(\chi_A)$.

As a last piece of notation, unless otherwise specified by a *probability Riesz tribe* is meant a pair $(\mathcal{T}, s)$ where $\mathcal{T}$ is a Riesz tribe and s is a σ-additive state. We recall that the measure that corresponds to s is, in this case, σ-additive.

2. Observables: definition and calculus

Probability measures on an MV-algebra are represented by states. It is natural to ask which is the notion that plays the same role with respect to random variables. Taking inspiration from the theory of quantum structures, let us define the notion of observable.

Definition 2.1. Let A be a σ-complete Riesz MV-algebra. For $\kappa \leq \omega$, a κ-dimensional *observable* on A is any σ-homomorphism of Riesz MV-algebras from $\mathrm{Borel}([0,1]^\kappa)$ to A, where $\mathrm{Borel}([0,1]^\kappa)$ denotes the Riesz tribe of Borel measurable functions from $[0,1]^\kappa$ to $[0,1]$.

In literature, observables are often defined in the finite dimensional case as functions from the σ-algebra of Borel sets of $\mathbb{R}^n$ to A, see e.g. [9, 10, 16, 24, 25, 27]. Another point of view is given in [8], where observables are defined over Borel measurable functions in $[0,1]^{\mathbb{R}^n}$. Our approach is different in spirit: our claim is that elements in $\mathrm{Borel}([0,1]^\kappa)$ are *generalized events*, since they represent equivalence classes of formulas in the logic $\mathcal{IR}$Ł, that naturally accommodate all operations needed to discuss probabilistic notions and to translate events from our natural language. In Sections 2.1 and 2.2, we analyze the interplay of our notion with the two different notions above-mentioned. It is worth noticing that Definition 2.1 could be introduced for an arbitrary, non necessarily countable, cardinal κ. Since formulas in $\mathcal{IR}$Ł have at most countably many propositional variables, and since our guiding idea is the fact that events are formulas in this logic, the definition is given for $\kappa \leq \omega$.

Proposition 2.2. *Let $\mathcal{X} : Borel([0,1]^\kappa) \to R$ be a κ-dimensional observable on $R \in \mathbf{RMV}_\sigma$. Then the algebra $Im(\mathcal{X}) = \{\mathcal{X}(a) \mid a \in Borel([0,1]^\kappa)\}$ is a κ-generated σ-complete Riesz MV-algebra that is a σ-subalgebra of R. Conversely, for any κ-generated σ-complete Riesz MV-algebra R, there exists an κ-dimensional observable $\mathcal{X}$ such that $R = Im(\mathcal{X})$.*

Proof. The result is a straightforward consequence of the definition of observable, the fact that $\mathrm{Borel}([0,1]^\kappa)$ is the free κ-generated algebra in $\mathbf{RMV}_\sigma$ and the fact that observables are homomorphisms having the free algebra as domain. □

The following result characterizes an observable via a uniquely defined measurable function, and it will be crucial in our development. The rest of the paper will be mainly focused on tribes, as partially justified by Proposition 1.1. A limited number of results will be also proved on continuous functions.

Theorem 2.3. *Let X be a nonempty set, $\mathcal{T} \subseteq [0,1]^X$ be a Riesz tribe and $f : X \to [0,1]^\kappa$ a measurable function w.r.t. $\mathcal{S}(\mathcal{T}) = \{A \subseteq X \mid \chi_A \in \mathcal{T}\}$. Then the function*

$$\mathcal{X}_f : \mathit{Borel}([0,1]^\kappa) \to \mathcal{T}, \quad \mathcal{X}_f(a) = a \circ f, \quad a \in \mathit{Borel}([0,1]^\kappa).$$

is a κ-dimensional observable on $\mathcal{T}$.
Conversely, for any κ-dimensional observable $\mathcal{X}$ on $\mathcal{T}$, there exists a unique $f : X \to [0,1]^\kappa$ such that $\mathcal{X} = \mathcal{X}_f$.

Proof. The fact that $\mathcal{X}_f$ is a homomorphism of Riesz MV-algebras is straightforward. Thus, we have to prove that it preserves countable joins. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in Borel$([0,1]^\kappa)$. We recall that countable suprema and infima are defined pointwise in Borel$([0,1]^\kappa)$ and in any tribe, therefore

$$\begin{aligned}\mathcal{X}_f\left(\bigvee_n g_n\right)(x) = \left(\bigvee_n g_n\right)(f(x)) = \sup_n\{g_n(f(x))\} = \sup_n\{(g_n \circ f)(x)\}\\ = \sup_n\{(\mathcal{X}_f(g_n))(x)\} = \left(\bigvee_n \mathcal{X}_f(g_n)\right)(x).\end{aligned}$$

Finally, let us prove that for any $a \in$ Borel$([0,1]^\kappa)$, $a \circ f \in \mathcal{T}$. By [22, Lemma 11.8(ii)], $\mathcal{T}$ is the tribe of all $\mathcal{S}(\mathcal{T})$-measurable functions, whence it is enough to prove that $a \circ f$ is $\mathcal{S}(\mathcal{T})$-measurable for any $a \in$ Borel$([0,1]^\kappa)$. This fact is easily deduced: For any $E \in \mathcal{B}([0,1])$, $(a \circ f)^{-1}(E) = f^{-1}(a^{-1}(E)) \in \mathcal{S}(\mathcal{T})$ since $f \in \mathcal{T}$ and $a \in$ Borel$([0,1]^\kappa)$.

Conversely, given an observable $\mathcal{X}$, we need to prove that there exists a function f that satisfies the claim. Let f be the function defined by $f = (f_i)_{i\in\kappa} : X \to [0,1]^\kappa$ with $f_i = \mathcal{X}(\pi_i)$. Using the fact that the projections form a generating set for Borel$([0,1]^\kappa)$, it is straightforward that $\mathcal{X} = \mathcal{X}_f$. Thus, we only need to prove that f is measurable w.r.t. $\mathcal{S}(\mathcal{T})$. Take $E = \Pi_{i=1}^\kappa E_i$, where each E_i is Borel subset of $[0,1]$. Such an E is a generator for $\mathcal{B}([0,1]^\kappa)$. Then,

$$\begin{aligned}f^{-1}(E) =& \{y \in X \mid (f_i(y))_{i\in\kappa} \in \Pi_{i=1}^\kappa E_i\} =\\ =& \{y \in X \mid f_i(y) \in E_i \text{ for all } i \in \kappa\} =\\ =& \{y \in X \mid y \in f_i^{-1}(E_i) \text{ for all } i \in \kappa\} = \bigcap_{i\in\kappa} f_i^{-1}(E_i).\end{aligned}$$

Since every $f_i \in \mathcal{T}$ is $\mathcal{S}(\mathcal{T})$-measurable and κ is countable, $\bigcap_{i\in\kappa} f_i^{-1}(E_i) \in \mathcal{S}(\mathcal{T})$ and f is $\mathcal{S}(\mathcal{T})$-measurable.

Finally, let $g : X \to [0,1]^\kappa$ be another function that satisfies the claim. Then, $\pi_i \circ g = \mathcal{X}(\pi_i) = \pi_i \circ f$, from which we deduce that for any $x \in X$, $g(x) = (\pi_i(g(x)))_{i\in\kappa} = (\pi_i(f(x)))_{i\in\kappa} = f(x)$. □

In the Introduction we have given ground motivation for the need of a new definition of random variable in logic. Theorem 2.3 provides yet another motivation for Definition 2.1 and for the study of these notions. Indeed, from a purely

nathematical point of view, we can look at random variables as measurable maps $\eta : (X, \Sigma, \mu) \to (E, \mathcal{E})$, from a probability space (X, Σ, μ) to a measurable space $(E, \mathcal{E})$. In our case, when the tribe $\mathcal{T}$ carries a σ-additive state s, the function f obtained in Theorem 2.3 is indeed a classical random variable, with $(E, \mathcal{E}) = ([0,1]^\kappa, \mathcal{B}([0,1]^\kappa))$ and (X, Σ, μ) being the space $(X, \mathcal{S}(\mathcal{T}), \mu_s)$, with $\mu_s : \mathcal{S}(\mathcal{T}) \to [0,1]$ given by $\mu_s(A) = s(\chi_A)$. Thus, from a more philosophical standpoint, we might think at the representation theorem as a sort of *layering of the uncertainty*: in writing $\mathcal{X}(a) = a \circ f$ we model at the same time the uncertainty given by probabilistic factors of our event, with the associate random variable f, as well as its fuzzy factors which are naturally codified in the formula a. To make this point more clear, let us consider the following example.

Example 2.4. Let $\mathcal{PC}$ be the Riesz tribe generated by piecewise constant functions in $[0,1]^X$, where $X = \{h, t\}$ is a 2-element set. Take $f \in \mathcal{PC}$ to be defined by

$$f(x) = \begin{cases} \alpha & x = h \\ \beta & x = t. \end{cases}$$

for $\alpha, \beta \in [0,1]$. We can think of such an f as the payoff, on head or tail, after the toss of a coin.

Let $\mathcal{X}\colon \mathrm{Borel}([0,1]) \to \mathcal{PC}$ be the observable defined defined as $\mathcal{X}_f$, that is, $\mathcal{X}(a) = a \circ f$ for any $a \in \mathrm{Borel}([0,1])$. In this point of view, the observable $\mathcal{X}$ gives more information on the toss of the coin: evaluating $\mathcal{X}$ on a Borel function help in modelling uncertainty on the toss itself. For example, $\mathcal{X}(id)$ can be thought as a fair toss in which the payoff are paid as stated. For $a = \chi_E$, with $\alpha \notin E$, we are modelling the case of a tosser that has bid himself on the game, betting on tail.

The following results show how we can "project" a κ-dimensional observable on each coordinate, and how we can "glue together" κ one-dimensional observables.

Proposition 2.5. *Let $\mathcal{X} : Borel([0,1]^\kappa) \to \mathcal{T}$ be a κ-dimensional observable over the Riesz tribe $\mathcal{T} \subseteq [0,1]^X$. The functions $\mathcal{X}_i : Borel([0,1]) \to \mathcal{T}$ defined by $\mathcal{X}_i = \mathcal{X}_{\pi_i \circ f}$ are one-dimensional observables over $\mathcal{T}$ for any $i \in \kappa$, where f is the measurable function given in Theorem 2.3.*

Proof. By Theorem 2.3 we only need to prove that each $\pi_i \circ f$ is an $\mathcal{S}(T)$-measurable function from X to $[0,1]$. By definition, $\pi_i \circ f$ is measurable w.r.t. $\mathcal{S}(\mathcal{T})$ if, and only if, for any $A \in \mathcal{B}([0,1])$, $(\pi_i \circ f)^{-1}(A) \in \mathcal{S}(\mathcal{T})$. Since projections are Borel-measurable and f is $\mathcal{S}(\mathcal{T})$–measurable, $f^{-1}(\pi_i^{-1}(A)) \in \mathcal{S}(\mathcal{T})$ and the claim is settled. □

Theorem 2.6 (Joint observable theorem)**.** *Given the Riesz tribe $\mathcal{T} \subseteq [0,1]^X$ and κ one-dimensional observables over $\mathcal{T}$, namely $\mathcal{X}_i : Borel([0,1]) \to \mathcal{T}$, with $i \in \kappa$, the joint function $\mathcal{J}_\kappa : Borel([0,1]^\kappa) \to \mathcal{T}$ defined by $\mathcal{J}_\kappa(a) = a \circ f$, with $f = (\mathcal{X}_i(id))_{i \in \kappa}$, is a κ-dimensional observable over $\mathcal{T}$, where $id : Borel([0,1]) \to Borel([0,1])$ is the identity function.*

Moreover, for any $i \in \kappa$ and any $a \in Borel([0,1])$, $\mathcal{J}_\kappa(a \circ \pi_i) = \mathcal{X}_i(a)$.

Proof. By Theorem 2.3, the map $\mathcal{J}_\kappa$ given in the claim is an observable if, and only if, f is an $\mathcal{S}(\mathcal{T})$–measurable function, and this can be argued exactly as done in the proof of Theorem 2.3. Finally, $\mathcal{J}_\kappa(a\circ\pi_i) = (a\circ\pi_i)\circ f = a\circ f_i = \mathcal{X}_i(a)$. $\square$

Using the Loomis-Sikorski theorem for Riesz MV-algebras, we obtain a representation theorem of any observable on a σ-complete Riesz MV-algebra, not necessarily a Riesz tribe, as well as a joint observable.

Theorem 2.7. *Let R be a σ-complete Riesz MV-algebra and let $(\mathcal{T}, X, \eta)$ be its tribe representation. For any κ-dimensional observable $\mathcal{X} : Borel([0,1]^\kappa) \to R$ there exists an $\mathcal{S}(\mathcal{T})$-measurable map $h : X \to [0,1]^\kappa$ such that, for any $a \in Borel([0,1]^\kappa)$,*

$$\mathcal{X}(a) = \eta(a \circ h).$$

Moreover, if $g : X \to [0,1]^\kappa$ is another function that satisfies the claim, then $\{x \in X \mid g(x) \neq h(x)\}$ is a meager set.

Proof. By surjectivity of η, for any projection $\pi_i \in \mathrm{Borel}([0,1]^\kappa)$, there exists a $h_i \in \mathcal{T}$ such that $\mathcal{X}(\pi_i) = \eta(h_i) \in R$. Let us prove that $h := (h_i)_{i\in\kappa} : X \to [0,1]^\kappa$ satisfies the claim.

The fact that such an h is $\mathcal{S}(\mathcal{T})$-measurable can be argued exactly as in the proof of Theorem 2.3. Whence, by Theorem 2.3, the map $\mathcal{X}_h$ defined by $a \in \mathrm{Borel}([0,1]^\kappa) \mapsto a \circ h$ is an observable on $\mathcal{T}$ such that $\mathcal{X}_h(\pi_i) = h_i$ for any $i \in \kappa$. It follows that $\eta \circ \mathcal{X}_h : \mathrm{Borel}([0,1]^\kappa) \to R$ is a κ-dimensional observable on R such that $(\eta \circ \mathcal{X}_h)(\pi_i) = \eta(\pi_i \circ h) = \eta(h_i) = \mathcal{X}(\pi_i)$ for any $i \in \kappa$. Since $\mathrm{Borel}([0,1]^\kappa)$ is generated by the projections, $\eta \circ \mathcal{X}_h = \mathcal{X}$.

Finally, let $g : X \to [0,1]^\kappa$ be another function such that $\eta(a \circ f) = \eta(a \circ g)$. In particular, for any $i \in \kappa$, we have $\eta(\pi_i \circ f) = \eta(\pi_i \circ g)$ and $\{x \in X \mid (\pi_i \circ f)(x) \neq (\pi_i \circ g)(x)\}$ is a meager set by the Loomis-Sikorski theorem. Since $f(x) \neq g(x)$ if there exists an $i \in \kappa$ such that $\pi_i(f(x)) \neq \pi_i(g(x))$, it follows that

$$\{x \in X \mid f(x) \neq g(x)\} \subseteq \bigcup_{i\in\kappa} \{x \in X \mid (\pi_i \circ f)(x) \neq (\pi_i \circ g)(x)\},$$

and it is, therefore, a meager set since κ is at most countable. $\square$

Proposition 2.8. *Let R be a σ-complete Riesz MV-algebra, $(\mathcal{T}, X, \eta)$ its tribe representation and $h : X \to [0,1]^\kappa$ an $\mathcal{S}(\mathcal{T})$-measurable function. Then the map $\mathcal{X}_h : Borel([0,1]^\kappa) \to R$ given by*

$$\mathcal{X}_h(a) = \eta(a \circ h), \qquad a \in Borel([0,1]^\kappa),$$

is a κ-dimensional observable on R.

Proof. By hypothesis, $h : X \to [0,1]^\kappa$ gives a κ-dimensional observable $\mathcal{Y}_h : \mathrm{Borel}([0,1]^\kappa) \to \mathcal{T}$. The claim now follows from the remark that $\mathcal{X}_h$ equals to the composition $\eta \circ \mathcal{Y}_h$. $\square$

Corollary 2.9 (Joint observable theorem)**.** *Given κ one-dimensional observables over $R \in \mathbf{RMV}_\sigma$, namely $\mathcal{X}_i : Borel([0,1]) \to R$, with $i \in \kappa$, there exists a κ-dimensional observable $\mathcal{J}_\kappa : Borel([0,1]^\kappa) \to R$ such that $\mathcal{J}_\kappa(a \circ \pi_i) = \mathcal{X}_i(a)$, for any $a \in Borel([0,1])$.*

Proof. Let $(\mathcal{T}, X, \eta)$ be the tribe representation of R and for any $i \in \kappa$, let us denote by h_i the $\mathcal{S}(\mathcal{T})$-measurable function, given in Theorem 2.7, such that $\mathcal{X}_i(a) = \eta(a \circ h_i)$.

Let $\mathcal{Y}_i : \mathrm{Borel}([0,1]) \to \mathcal{T}$ be the observables defined by $\mathcal{Y}_i(a) = a \circ h_i$, and let $\mathcal{H}_\kappa : \mathrm{Borel}([0,1]^\kappa) \to \mathcal{T}$ be the joint observable of the $\mathcal{Y}_i$'s. It follows that $\mathcal{H}_\kappa(a) = a \circ (h_i)_{i\in\kappa}$ for any $a \in \mathrm{Borel}([0,1]^\kappa)$ and that $h := (h_i)_{i\in\kappa}$ is $\mathcal{S}(\mathcal{T})$-measurable. By Proposition 2.8, $\mathcal{J}_\kappa(a) := \eta(a \circ h)$ is an n-dimensional observable on R. For any $i \in \kappa$ and any $a \in \mathrm{Borel}([0,1])$, $\mathcal{J}_\kappa(a \circ \pi_i) = \eta((a \circ \pi_i) \circ h) = \eta(a \circ h_i) = \mathcal{X}_i(a)$, settling the claim. □

In Definition 2.1 we defined an observable to be any homomorphism from the free algebra in $\mathbf{RMV}_\sigma$ to a generic σ-complete Riesz MV-algebra, while Theorems 2.3 and 2.7 give a clear-cut representation of observables in the case of tribes, and a representation up to meager sets in general. Nonetheless, the representation theorem holds for algebras of almost everywhere equal measurable functions, up to measure-null sets.

Notation. For any probability Riesz tribe $(\mathcal{T}, s)$, in what follows we shall denote by $\mathcal{T}_s$ the Riesz MV-algebra obtained from $\mathcal{T}$ identifying μ_s-equal everywhere functions with respect to the probability space $(X, \mathcal{S}(\mathcal{T}), \mu_s)$. We also remark that, since functions in $\mathcal{T}_s$ are bounded and X is of finite measure, integrals of functions in $\mathcal{T}_s$ are finite.

Lemma 2.10. *The map $[\cdot]_s : \mathcal{T} \to \mathcal{T}_s$, that assigns to each function its equivalent class, is a σ-homomorphism. As a consequence, in the σ-complete Riesz MV-algebra $\mathcal{T}_s$ countable joins are defined pointwisely on the representative of the class.*

Proof. It follows from [19, Example 23.3(iv)] that the set of all real μ_s-almost everywhere finite-valued and μ_s-measurable functions on $(X, \mathcal{S}(\mathcal{T}), \mu_s)$ is a super Dedekind complete Riesz space that contains $\mathcal{T}_s$. Take any countable set $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}_s$. We need to prove that the supremum of the set is pointwise. Since each function in $\mathcal{T}$ is bounded by $\mathbf{1}$, the function identically equal to one, we are in the hypothesis of the first case of [19, Example 23.3(iv)]. Therefore, there exists a countable subset $D = \{f_{n_k}\}_{k\in\mathbb{N}}$ such that $\bigvee_n f_n$ coincides with the pointwise supremum of the functions in D. Let us denote by g the pointwise supremum of $\{f_n\}_{n\in\mathbb{N}}$, that is, $g(x) = \sup_n f_n(x)$. Since $f_n(x) \leq (\bigvee_n f_n)(x)$, it follows that $g \leq \bigvee_n f_n$. On the other end, $(\bigvee_n f_n)(x) = \sup_k f_{n_k}(x) \leq \sup_n f_n(x) = g(x)$, which entails $\bigvee_n f_n = g$. Finally, we remark that $\bigvee_n f_n \leq \mathbf{1}$ and therefore it belongs to $\mathcal{T}_s$. □

Corollary 2.11. *With the above-defined notations, for any measurable function $f : X \to [0,1]^\kappa$ w.r.t. $\mathcal{S}(\mathcal{T}) = \{A \subseteq X \mid \chi_A \in \mathcal{T}\}$, the function*

$$\mathcal{X}_f : \mathit{Borel}([0,1]^\kappa) \to \mathcal{T}_s, \quad \mathcal{X}_f(a) = [a \circ f]_s, \quad a \in \mathit{Borel}([0,1]^\kappa).$$

is a κ-dimensional observable on $\mathcal{T}_s$.
Conversely, for any κ-dimensional observable $\mathcal{X}$ on $\mathcal{T}$, there exists a unique (up to μ_s-null sets) measurable function $f : X \to [0,1]^\kappa$ such that $\mathcal{X} = \mathcal{X}_f$.

Proof. If $f : X \to [0,1]^\kappa$ is $\mathcal{S}(\mathcal{T})$-measurable, it is easy to check that $\mathcal{X}_f$ is an observable. In particular, Lemma 2.10 ensures that $\mathcal{X}(\bigvee_n a_n) = \bigvee_n \mathcal{X}(a_n)$. Conversely, define $f = (f_i)_{i\in\kappa}$ by choosing $f_i \in [\mathcal{X}(\pi_i)]_s$. The measurability of f and the fact that $\mathcal{X} = \mathcal{X}_f$ can be proved as in the proof of Theorem 2.3. Let $g = (g_i)_{i\in\kappa}$ be another measurable function that satisfies the claim. Then $\mathcal{X}_f(\pi_i) = \mathcal{X}(\pi_i) = \mathcal{X}_g(\pi_i)$, that is $[f_i]_s = [g_i]_s$ for any $i \in \kappa$. Thus, $\{x \in X \mid f(x) \neq g(x)\} \subseteq \bigcup_{i\in\kappa}\{x \in X \mid f_i(x) \neq g_i(x)\}$. Since $\mu_s(\{x \in X \mid f_i(x) \neq g_i(x)\}) = 0$ for any $i \in \kappa$, it follows that $f = g$ almost everywhere. □

To close this section, we present our version of the *calculus* of observables. The key idea is that, although observables do not form an algebra in themselves (in general, homomorphisms cannot be endowed with pointwise defined operations), we can define the Łukasiewicz operations on one-dimensional observables via their joint observable.

Proposition 2.12. *Let $\mathcal{X}_i \colon \mathit{Borel}([0,1]) \to R$ be one-dimensional observables $R \in \mathbf{RMV}_\sigma$, for $i \in \kappa$. Take $\phi \in \mathit{Borel}([0,1]^\kappa)$. Then the following hold:*

(i) *Let $\mathcal{J}_\kappa$ be the joint observable of the $\mathcal{X}_i$'s. Then $\phi \bullet \mathcal{J}_\kappa := \mathit{Borel}([0,1]) \to R$, defined by $a \mapsto \mathcal{J}_\kappa(a \circ \phi)$ is a well-defined one-dimensional observable.*

(ii) *If $f = (f_i)_{i\in\kappa}$ and $(\phi \circ f)(x) := \phi((f_i(x))_{i\in\kappa})$, for $f_i \in \mathit{Borel}([0,1])$ such that $\mathcal{X}_i(a) = \eta(a \circ f_i)$, then $\phi \bullet \mathcal{J}_\kappa = \mathcal{X}_{\phi\circ f}$.*

Proof. Let $(\mathcal{T}, X, \eta)$ be the tribe representation of R.

(i) By our choice of the function ϕ, it follows that $a \circ \phi \in \mathrm{Borel}([0,1]^\kappa)$ for any $a \in \mathrm{Borel}([0,1])$. Whence, $\phi \bullet \mathcal{J}_\kappa$ is a one-dimensional observable because it is the composition of $\mathcal{J}_\kappa : \mathrm{Borel}([0,1]^\kappa) \to R$ and $\mathcal{X}_\phi : \mathrm{Borel}([0,1]) \to \mathrm{Borel}([0,1]^\kappa)$.

(ii) We first remark that $\phi \circ f : X \to [0,1]$ is an $\mathcal{S}(\mathcal{T})$-measurable function. Indeed, $f = (f_i)_{i\in\kappa}$ has been proved to be $\mathcal{S}(\mathcal{T})$-measurable in Theorem 2.3, while ϕ belongs to $\mathrm{Borel}([0,1]^\kappa)$ by hypothesis.

For any $a \in \mathrm{Borel}([0,1])$ and any $x \in X$,

$$(a \circ (\phi \circ f))(x) = a(\phi(f_i(x))_{i\in\kappa}) = ((a \circ \phi) \circ f)(x).$$

Then, $(a \circ (\phi \circ f)) = ((a \circ \phi) \circ f) : X \to [0,1]$ and $\mathcal{X}_{\phi\circ f}(a) = \eta(a \circ (\phi \circ f)) = \eta((a \circ \phi) \circ f) = \mathcal{J}_\kappa(a \circ \phi) = (\phi \bullet \mathcal{J}_\kappa)(a)$. □

As an example, we denote by $\phi(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ the observable given by $\phi \bullet \mathcal{J}_n$. Whence, $\mathcal{X}_1 \oplus \mathcal{X}_2$ is the observable given by $\phi(x, y) = \min(x + y, 1)$.

2.1. The relation with unbounded observables

As mentioned before, in [8], observables on σ-complete MV-algebras are defined as σ-complete homomorphisms of MV-algebras $\mathcal{X} : \mathrm{Borel}(\mathbb{R}^n) \to M$, where M is a σ-complete MV-algebra and $\mathrm{Borel}(\mathbb{R}^n) \subseteq [0,1]^{\mathbb{R}^n}$ is the tribe of all Borel-measurable functions on $\mathbb{R}^n$. Moreover, it was proved that M admits an observable if, and only if, M is weakly divisible, that is, for any $n \in \mathbb{N}$ there exists $v \in M$ such that $nv = 1$. We note that any weakly divisible σ-complete MV-algebra is divisible, so it is a Riesz MV-algebra.

Following the same ideas, we can define *unbounded* observables on a σ-complete Riesz MV-algebras as σ-homomorphisms $\mathcal{X} : \mathrm{Borel}(\mathbb{R}^n) \to R$, where we look at $\mathrm{Borel}(\mathbb{R}^n)$ as a Riesz tribe.

Since any MV-homomorphism between semisimple Riesz MV-algebras is a Riesz MV-homomorphism, we deduce the validity of all results of [8] in the case of Riesz MV-algebras. We also point out that many of the results in [8] either use tribes that contain constant fuzzy sets, or they require for the MV-algebra considered to be weakly divisible, and both conditions are trivially satisfied by Riesz MV-algebras and Riesz tribes.

Giving a precise statement for these remarks, we can write the following lemma.

Lemma 2.13. *For $R \in \mathbf{RMV}_\sigma$, any homomorphism of σ-complete Riesz MV-algebras $h : \mathit{Borel}(\mathbb{R}^n) \to \mathit{Borel}([0,1]^n)$ and any observable $\mathcal{X} : \mathit{Borel}([0,1]^n) \to R$, the composition $\mathcal{X} \circ h : \mathit{Borel}(\mathbb{R}^n) \to R$ defines an observable in the sense of [8].*

Proof. It is a straightforward application of the fact that $\mathrm{Borel}(\mathbb{R}^n)$ is a Riesz tribe and that both h and $\mathcal{X}$ are homomorphisms in $\mathbf{RMV}_\sigma$. □

Example 2.14 (Standard generalization). For any observable $\mathcal{X} : \mathrm{Borel}([0,1]^n) \to R$, we define its standard generalization as the MV-observable $\hat{\mathcal{X}} : \mathrm{Borel}(\mathbb{R}^n) \to R$, $\hat{\mathcal{X}}(a) = \mathcal{X}(a \mid_{[0,1]^n})$. Indeed, the map $\phi : a \in \mathrm{Borel}(\mathbb{R}^n) \mapsto a \mid_{[0,1]^n} \in \mathrm{Borel}([0,1]^n)$ satisfies the condition of Lemma 2.13: It is easy to show that $\phi(a) \in \mathrm{Borel}([0,1]^n)$ for any $a \in \mathrm{Borel}(\mathbb{R}^n)$ and, using the fact that all operations between Borel functions are defined pointwise (including the countable suprema), ϕ is a homomorphism of σ-complete MV-algebras.

To give a converse of Lemma 2.13, let us consider the class $\mathcal{M}$ of those σ-complete MV-algebras that are the MV-reduct of a σ-complete Riesz MV-algebra, in symbols,

$$\mathcal{M} = \{M \in \mathbf{MV}_\sigma \mid M \text{ is the MV-reduct of some algebra in } \mathbf{RMV}_\sigma\}.$$

Notice that it is not always that case that $A \in \mathbf{MV}_\sigma$ is the reduct of a σ-complete Riesz MV-algebra. Indeed, an MV-tribe that does not contain the constant functions cannot be a Riesz tribe.

Lemma 2.15. *For any MV-observable (in the sense of [8]) $\mathcal{Y} : \mathit{Borel}(\mathbb{R}^n) \to M$, with $M \in \mathcal{M}$, and any Borel-measurable function $h : \mathbb{R}^n \to [0,1]^n$, the map*

$\mathcal{X}$: *Borel*$([0,1]^n) \to M$ *defined by* $\mathcal{X}(a) = \mathcal{Y}(a \circ h)$ *is an observable on* M*, endowed with the structure of Riesz MV-algebra.*

Proof. It is straightforward to verify that for any $f \in \mathrm{Borel}([0,1]^n)$, $f \circ h \in \mathrm{Borel}(\mathbb{R}^n)$. Thus, since pre-composition is stable under all operations and since each $M \in \mathcal{M}$ can be endowed with the structure of Riesz MV-algebra, $\mathcal{X}$ is a well-defined observable in the sense of Definition 2.1. □

These remarks allow us to obtain an unbounded joint observable. Indeed, let us consider n one-dimensional observables over $R \in \mathbf{RMV}_\sigma$, namely $\mathcal{X}_1, \ldots, \mathcal{X}_n : \mathrm{Borel}([0,1]) \to R$, and their standard generalization $\widehat{\mathcal{X}}_1, \ldots, \widehat{\mathcal{X}}_n : \mathrm{Borel}(\mathbb{R}) \to R$ as given by Example 2.14.

Corollary 2.16. *The joint MV-observable (in the sense of [8])* $\mathcal{Y}$: *Borel*$(\mathbb{R}^n) \to R$ *of* $\widehat{\mathcal{X}}_1, \ldots, \widehat{\mathcal{X}}_n$ *always exists. Moreover, for any Borel-measurable function* $h : \mathbb{R}^n \to [0,1]^n$*, the map* $\mathcal{X}$: *Borel*$([0,1]^n) \to R$ *defined by* $\mathcal{X}(a) = \mathcal{Y}(a \circ h)$ *is an observable on* R*.*

Proof. Since R is a σ-complete Riesz MV-algebra, we are in the hypothesis of [8, Theorem 3.3] and therefore the joint MV-observable $\mathcal{Y} : \mathrm{Borel}(\mathbb{R}^n) \to R$ always exists and it satisfies the condition $\hat{\mathcal{X}}_i(a) = \mathcal{Y}(a \circ \overline{\pi_i})$, where $\overline{\pi_i} : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection. The second part of the claim is a consequence of Lemma 2.15. □

Finally, if $\mathcal{X}$ is any observable on R, in particular if it is unbounded, we can define the spectrum of $\mathcal{X}$ as

$$\sigma(\mathcal{X}) = \bigcap \{C \subseteq \mathbb{R}^n \mid \mathcal{X}(\chi_C) = 1,\ C \text{ closed subset}\}.$$

Then, $\mathcal{X}$ is called bounded if $\sigma(\mathcal{X})$ is a compact subset of $\mathbb{R}^n$. Thus, we might think of Definition 2.1 as a restriction of the case of (unbounded) observables admitting $[0,1]^n$ as spectrum.

2.2. The relation with observables defined over Borel subsets

In [9, 10, 16, 24, 25, 27], observables are defined over Borel subsets rather than Borel functions. Let us now explore the relation of this definition with our notion.

Let R be a σ-complete Riesz MV-algebra. Let us set $\Sigma([0,1]^n)) = \{\chi_A \colon A \in \mathcal{B}([0,1]^n)\}$, thus $\Sigma([0,1]^n))$ has a natural structure of a σ-complete Boolean algebra.

A map $\mathcal{X} : \Sigma([0,1]^n) \to R$ is said to be an *n-dimensional weak observable* if

(i) $\mathcal{X}(\chi_{[0,1]^n}) = 1$

(ii) $\mathcal{X}(\chi_A) + \mathcal{X}(\chi_B)$ is defined whenever $A \cap B = \emptyset$, $A, B \in \mathcal{B}([0,1]^n)$, and in this case $\mathcal{X}(\chi_A \vee \chi_B) = \mathcal{X}(\chi_A) + \mathcal{X}(\chi_B)$,

(iii) if $\{A_k\} \nearrow A$, then $\bigvee_k \mathcal{X}(\chi_{A_k}) = \mathcal{X}(\chi_A)$, where $A_k, A \in \mathcal{B}([0,1]^n)$ for $k \geq 1$.

Clearly, if $\mathcal{X} : \mathrm{Borel}([0,1]^n) \to R$ is an n-dimensional observable, then the restriction of $\mathcal{X}$ onto $\Sigma([0,1]^n)$ is an n-dimensional weak observable.

The converse statement is contained in the following result. It is worth noticing that, to make all computations smoother, the proof of this result is obtained by working in the setting of Riesz spaces, and then restricting attention again to Riesz MV-algebras. We also remark that, $\Sigma([0,1]^n)$ being a Boolean algebra, it carries a natural structure of MV-algebra, upon setting $\oplus = \vee$.

Theorem 2.17. *Let $\mathcal{X}\colon \Sigma([0,1]^n) \to R$ be an n-dimensional weak observable. Then $\mathcal{X}$ can be extended to a unique n-dimensional observable $\widehat{\mathcal{X}}$ if, and only if, $\mathcal{X}$ is an MV-σ-homomorphism from $\Sigma([0,1]^n)$ into R.*

Proof. For the non-trivial direction, assume that $\mathcal{X}$ is an MV-σ-homomorphism. Without loss of generality, by Proposition 1.1, we can assume $R = \Gamma(C(X), 1_X)$ for some compact, Hausdorff, basically disconnected space X. In what follows we shall work directly on the Riesz space $C(X)$ rather than on its unit interval R.

Let $a \in \mathrm{Borel}([0,1]^n)$ be a simple function, so that $a = \sum_{i=1}^k \alpha_i \chi_{E_i}$, where $E_1, \ldots, E_k$ are mutually disjoint Borel subsets from $[0,1]^n$. Then we define

$$\widehat{\mathcal{X}}(a) := \sum_{i=1}^k \alpha_i \mathcal{X}(\chi_{E_i}). \tag{1}$$

First we show that $\widehat{\mathcal{X}}$ is well-defined. Thus, let $a = \sum_{j=1}^m \beta_j \chi_{F_j}$, where $F_1, \ldots, F_m \in \mathcal{B}([0,1]^n)$ are mutually disjoint. Then $a = \sum_{i=1}^k \sum_{j=1}^m \gamma_{ij} \chi_{E_i \cap F_j}$, where $\alpha_i = \gamma_{ij} = \beta_j$ if $E_i \cap F_j \neq \emptyset$ and $\gamma_{ij} = 0$ otherwise, so that

$$\sum_{i=1}^k \alpha_i \mathcal{X}(\chi_{E_i}) = \sum_{i=1}^k \sum_{j=1}^m \gamma_{ij} \mathcal{X}(\chi_{E_i \cap F_j}) = \sum_{j=1}^m \beta_j \mathcal{X}(\chi_{F_j}),$$

which shows that $\widehat{\mathcal{X}}$ is well-defined by (1) for simple functions. In addition, if $a : [0,1]^n \to \mathbb{R}$ is a simple Borel measurable function not necessarily in $\mathrm{Borel}([0,1]^n)$, the definition of $\widehat{\mathcal{X}}$ given by (1) still applies. Then, with an argument similar to the one commonly used for integrals, we have that $\widehat{\mathcal{X}}(a+b) = \widehat{\mathcal{X}}(a) + \widehat{\mathcal{X}}(b)$ and $\widehat{\mathcal{X}}(\alpha a) = \alpha \widehat{\mathcal{X}}(a)$. Hence, on the class of simple functions in $\mathrm{Borel}([0,1]^n)$, $\widehat{\mathcal{X}}$ is additive, homogeneous, and monotone.

If $a = \sum_{i=1}^k \alpha_i \chi_{A_i}$ and $b = \sum_{j=1}^m \beta_j \chi_{B_j}$, it follows that

$$a + b = \sum_{i,j} (\alpha_i + \beta_j) \chi_{A_i \cap B_j} \text{ and } a \oplus b = (a+b) \wedge 1_{[0,1]^n} = \sum_{i,j} \gamma_{ij} \chi_{A_i \cap B_j},$$

where $\gamma_{ij} = \alpha_i + \beta_j$ if $\alpha_i + \beta_j \leq 1$ and $\gamma_{ij} = 1$ if $\alpha_i + \beta_j > 1$. Consequently, it is possible to show that

$$\widehat{\mathcal{X}}(a \oplus b) = (\widehat{\mathcal{X}}(a) + \widehat{\mathcal{X}}(b)) \wedge 1_X = \widehat{\mathcal{X}}(a) \oplus \widehat{\mathcal{X}}(b).$$

Moreover, if a, b are simple functions in Borel$([0,1]^n)$, it is possible, with an argument similar to the one given for $\oplus$, to show that $\widehat{\mathfrak{X}}(a \wedge b) = \widehat{\mathfrak{X}}(a) \wedge \widehat{\mathfrak{X}}(b)$ and $\widehat{\mathfrak{X}}(a \vee b) = \widehat{\mathfrak{X}}(a) \vee \widehat{\mathfrak{X}}(b)$.

Claim. *Let $\{a_i\}_i$ be a sequence of simple functions in Borel$([0,1]^n)$ such that $\{a_i(t)\}_i \nearrow a(t)$, $t \in [0,1]^n$, where $a \in$ Borel$([0,1]^n)$ is also a simple function. Then $\{\widehat{\mathfrak{X}}(a_i)\}_i \nearrow \widehat{\mathfrak{X}}(a)$.*

Proof. Since $\hat{\mathfrak{X}}$ is monotone on simple functions, it is clear that $\lim_i \widehat{\mathfrak{X}}(a_i) =: \hat{a} \leq \widehat{\mathfrak{X}}(a)$. We shall now prove the converse inequality. Let $\epsilon > 0$ be given and we define

$$\begin{aligned} A &= \{t \in [0,1]^n \colon a(t) \neq a_1(t)\}, \\ A_i &= \{t \in A \colon a(t) > \epsilon + a_i(t)\}, \quad i = 1, 2, \ldots. \end{aligned}$$

Then $\{A_i\}_i \searrow \emptyset$ so that $\{\mathfrak{X}(\chi_{A_i})\}_i \searrow 0_X$ because $\mathfrak{X}$ is a weak n-dimensional observable. We denote by $K = \max(a - a_1)$. Then

$$\begin{aligned} \widehat{\mathfrak{X}}(a) &= \widehat{\mathfrak{X}}(a - a_i) + \widehat{\mathfrak{X}}(a_i) = \widehat{\mathfrak{X}}((a - a_i)\chi_{A_i}) + \widehat{\mathfrak{X}}((a - a_i)\chi_{A \setminus A_i}) + \widehat{\mathfrak{X}}(a_i) \\ &\leq K\widehat{\mathfrak{X}}(\chi_{A_i}) + \epsilon\mathfrak{X}(\chi_{A \setminus A_i}) + \widehat{\mathfrak{X}}(a_i) \leq K\mathfrak{X}(\chi_{A_i}) + \epsilon\mathfrak{X}(\chi_A) + \widehat{\mathfrak{X}}(a_i). \end{aligned}$$

If $i \to \infty$, we obtain $\widehat{\mathfrak{X}}(a) \leq \epsilon\mathfrak{X}(\chi_A) + \lim_i \widehat{\mathfrak{X}}(a_i)$. Since $\epsilon > 0$ is arbitrary, we have $\widehat{\mathfrak{X}}(a) \leq \lim_i \widehat{\mathfrak{X}}(a_i)$, which finishes the proof of the claim. We remark that, since $\hat{a}$ is a continuous function, the limit is uniform. Moreover, the uniform limit of a non-decreasing sequence of continuous functions coincides with its lattice supremum. □

Now, let $a \in$ Borel$([0,1]^n)$ be an arbitrary function. There is a non-decreasing sequence $\{a_i\}_i$ of simple functions in Borel$([0,1]^n)$ such that $\lim_i a_i(t) = a(t)$, $t \in [0,1]^n$. Then $\{\widehat{\mathfrak{X}}(a_i)\}_i$ is a non-decreasing sequence of continuous functions in $\Gamma(C(X), 1_X)$. There is a continuous function $\hat{a} \in \Gamma(C(X), 1_X)$ such that $\bigvee_i \widehat{\mathfrak{X}}(a_i)_i = \hat{a}$. Then we define

$$\widehat{\mathfrak{X}}(a) := \bigvee_i \mathfrak{X}(a_i) = \hat{a}. \tag{2}$$

We have to show that (2) does not depend on the choice of $\{a_i\}_i$. So let $\{b_i\}_i$ be another sequence of simple functions from Borel$([0,1]^n)$ such that $\{b_i(t)\}_i \nearrow a(t)$ for each $t \in [0,1]^n$.

Let $j \geq 1$ be a fixed integer. For each $i \geq 1$, we define $c_i = a_j \wedge b_i$. Then $\{c_i(t)\}_i \nearrow a_j(t)$, $t \in [0,1]^n$, which by Claim gives

$$\widehat{\mathfrak{X}}(a_j) = \lim_i \widehat{\mathfrak{X}}(c_i) \leq \lim_i \widehat{\mathfrak{X}}(b_i),$$

and therefore, $\lim_j \widehat{\mathfrak{X}}(a_j) \leq \lim_i \widehat{\mathfrak{X}}(b_i)$. If we exchange the role of a_i and b_j, we obtain $\lim_i \widehat{\mathfrak{X}}(b_i) \leq \lim_j \widehat{\mathfrak{X}}(a_j)$ which proves that $\lim_j \widehat{\mathfrak{X}}(a_j) = \lim_i \widehat{\mathfrak{X}}(b_i)$. Hence, $\hat{a} = \bigvee_i \mathfrak{X}(b_i)$, so that (2) is well-defined.

Now, if $a, b \in \mathrm{Borel}([0,1]^n)$, and if $\{a_i\}_i \nearrow a$ and $\{b_i\}_i \nearrow b$ for two sequences of simple functions in $\mathrm{Borel}([0,1]^n)$, we have $\{a_i \wedge b_i\}_i \nearrow a \wedge b$, so that

$$\begin{aligned}\widehat{\mathcal{X}}(a \wedge b) &= \bigvee_i \widehat{\mathcal{X}}(a_i \wedge b_i) = \bigvee_i (\widehat{\mathcal{X}}(a_i) \wedge \widehat{\mathcal{X}}(b_i)) = \bigvee_{ij} (\widehat{\mathcal{X}}(a_i) \wedge \widehat{\mathcal{X}}(b_j)) = \\ &= \bigvee_i \widehat{\mathcal{X}}(a_i) \wedge \bigvee_j \widehat{\mathcal{X}}(b_j) = \widehat{\mathcal{X}}(a) \wedge \widehat{\mathcal{X}}(b).\end{aligned}$$

In addition, $\widehat{\mathcal{X}}(1_{[0,1]^n} - a) = \lim_i \widehat{\mathcal{X}}(1_{[0,1]^n} - a_i) = 1_X - \lim_i \widehat{\mathcal{X}}(a_i)$. Therefore, $\widehat{\mathcal{X}}(a \vee b) = \widehat{\mathcal{X}}(a) \vee \widehat{\mathcal{X}}(b)$. Moreover, $\widehat{\mathcal{X}}$ can be extended to all bounded Borel measurable functions $a : [0,1]^n \to \mathbb{R}$, so that $\widehat{\mathcal{X}}$ is additive, homogeneous and monotone.

Finally, let $\{a_i(t)\}_i \nearrow a(t)$, $t \in [0,1]^n$, where $a, a_1, a_2, \ldots$ are arbitrary functions in $\mathrm{Borel}([0,1]^n)$. By monotonicity of $\widehat{\mathcal{X}}$ we have $\bigvee_i \widehat{\mathcal{X}}(a_i) \leq \widehat{\mathcal{X}}(a)$. For each $i \geq 1$, there is a sequence $\{a_{ij}\}_j$ of simple functions such that $\{a_{ij}(t)\}_j \nearrow a_i(t)$ for each $t \in [0,1]^n$. Define new simple functions $\hat{a}_{ij} = a_{1j} \vee a_{2j} \vee \cdots \vee a_{ij}$ for all $i, j \geq 1$. Whence, $a_{ij} \leq \hat{a}_{ij}$ and $\hat{a}_{ij} \leq \hat{a}_{ii}$ if $j \leq i$ and $\hat{a}_{ij} \leq \hat{a}_{jj}$ if $j > i$, and $a_{kk} \leq a_{k+1,k+1}$ for each $k \geq 1$. Then $\sup_{ij} a_{ij}(t) = \sup_k \hat{a}_{kk}(t) = \lim_k \hat{a}_{kk}(t) = a(t)$, $t \in [0,1]^n$, so that $\bigvee_k \widehat{\mathcal{X}}(\hat{a}_{kk}) = \widehat{\mathcal{X}}(a)$. This yields

$$\bigvee_i \widehat{\mathcal{X}}(a_i) = \bigvee_{i,j} \widehat{\mathcal{X}}(\hat{a}_{ij}) = \bigvee_k \widehat{\mathcal{X}}(\hat{a}_{kk}) = \widehat{\mathcal{X}}(a),$$

which concludes the proof that $\widehat{\mathcal{X}}$ is an n-dimensional observable on $\mathrm{Borel}([0,1]^n)$ which is an extension of $\mathcal{X}$. In addition, from the construction we conclude that $\widehat{\mathcal{X}}$ is a unique n-dimensional observable that is an extension of the weak n-dimensional observable $\mathcal{X}$. □

To close this subsection, let us give an example of a weak n-dimensional observable which is not an n-dimensional observable. Let $\{a_i\}_i$ be a finite or infinite sequence of elements of a $\Gamma(R, u)$ such that $\sum_i a_i = u$ and let $\{t_i\}_i$ be a sequence of mutually different numbers of $\mathbb{R}^n$. Set

$$\mathcal{X}(\chi_A) = \sum \{a_i \colon t_i \in A\}, \quad \chi_A \in \Sigma([0,1]^n). \tag{3}$$

Then $\mathcal{X}$ is an example of a weak n-dimensional observable on $\Gamma(R, u)$.

Take the Riesz MV-algebra $\Gamma(\mathbb{R}, 1)$, $a_1 = 0.3$, $a_2 = 0.7$, $t_1 = 0.3$, $t_2 = 0.4$. Define a weak one-dimensional observable $\mathcal{X}$ by (3). If $E = \{0.3\}$ and $F = \{0.4\}$, then $\mathcal{X}(\chi_E) = 0.3$, $\mathcal{X}(\chi_F) = 0.7$, and $0 = \mathcal{X}(\chi_E \wedge \chi_F) < \mathcal{X}(\chi_E) \wedge \mathcal{X}(\chi_F) = 0.3$, so $\mathcal{X}$ is not a one-dimensional observable.

3. A spectral representation

In this section, we give a representation theorem that is inspired by the spectral resolution of a Hilbert space. The first step is the following definition.

Definition 3.1. Let R be a σ-complete Riesz MV-algebra and let $(\mathcal{T}, X, \eta)$ be its tribe representation. For any $r \in R$, let $f_r \in \mathcal{T} \subseteq [0,1]^X$ be a function such that $\eta(f_r) = r$ and let $\lambda_r : \mathrm{Borel}([0,1]) \to R$ to be the observable defined by $\lambda_r(a) = \eta(a \circ f_r)$.

Notice that an application of Theorem 2.7 implies that the assignment λ_r given in Definition 3.1 is indeed a well-defined one-dimensional observable.

Let us denote by $\mathcal{O}_\kappa(R)$ the set of all κ-dimensional observables on the algebra $R \in \mathbf{RMV}_\sigma$.

Proposition 3.2. *Let $R \in \mathbf{RMV}_\sigma$. The function $\Phi : R \to \mathcal{O}_1(R)$ defined by $r \mapsto \lambda_r$ is an injective map such that $\lambda_{\mathbf{1}_R}(a) = \eta(a(1))$ for any $a \in$ Borel$([0,1])$ and it preserves $\oplus$ and $\neg$, that is,*

$$\lambda_r \oplus \lambda_s = \lambda_{r \oplus s}, \qquad \neg\lambda_r = \lambda_{\neg r}.$$

Moreover, if $R = \mathcal{T} \subseteq [0,1]^X$ is a Riesz tribe, the following holds. Let $a \in$ Borel$([0,1])$ be a monotone invertible function and $\{f_k\}_k$ be a sequence of elements from $\mathcal{T}$. Set $f = \bigvee_k f_k$ and $g = \bigwedge_k f_k$. If a is non-decreasing, then $\bigvee_k \lambda_{f_k}(a) = \lambda_f(a)$ and $\bigwedge_k \lambda_{f_k}(a) = \lambda_g(a)$. If a is non-increasing, then $\bigvee_k \lambda_{f_k}(a) = \lambda_g(a)$ and $\bigwedge_k \lambda_{f_k}(a) = \lambda_f(a)$.

Proof. Let r, s be distinct elements of R. Then $\lambda_r(id) = \eta(id \circ f_r) = \eta(f_r) = r \neq s = \eta(f_s) = \eta(id \circ f_s) = \lambda_s(id)$, whence $\lambda_r \neq \lambda_s$ and Φ is injective. To prove that Φ commutes with the MV-operations, it is enough to notice that both the cases of $\oplus$ and $\neg$ follow from Proposition 2.12 with $\phi(x,y) = \min(x+y, 1)$ and $\phi(x) = 1 - x$ respectively. If $r = \mathbf{1}_R$, $a \in \mathrm{Borel}([0,1])$ and $x \in X$, $\lambda_{1_R}(a)(x) = \eta(a(1_R(x))) = \eta(\mathbf{a}(\mathbf{1}))$, where $\mathbf{a}(\mathbf{1})$ is the function that takes constantly the value $a(1)$.

Let $a \in \mathrm{Borel}([0,1])$ be invertible. Then a maps injectively $[0,1]$ onto $[0,1]$ and a is either strictly increasing or strictly decreasing. In addition, if a is strictly increasing (decreasing), so is its inverse a^{-1}.

Let $\{f_k\}_k$ be a sequence of functions in $\mathcal{T}$. Assume that a is strictly increasing. It is easily seen that $a \circ \bigvee_k f_k$ is an upper bound for $\{a \circ f_k\}_k$, therefore, $\bigvee_k (a \circ f_k) \leq a \circ \bigvee_k f_k$. Let $h \in \mathcal{T}$ be another upper bound for $\{f_k\}$, that is for any $k \in \mathbb{N}$, $a \circ f_k \leq h$. By hypothesis, $f_k \leq a^{-1} \circ h$, whence $\bigvee_k f_k \leq a^{-1} \circ h$, and again $a \circ \bigvee_k f_k \leq h$. From the latter inequality we deduce that $a \circ \bigvee_k f_k = \bigvee_k (a \circ f_k)$.

In the same way we prove the other three inequalities.

□

The map Φ and the observables λ_r will be subsequently called a *spectral representation* of R.

Corollary 3.3 (Joint spectral observable)**.** *Let $\kappa \leq \omega$, R a σ-complete Riesz MV-algebra, $\{\lambda_r\}$ its spectral representation and $(\mathcal{T}, X, \eta)$ its tribe representation. For any $\mathbf{r} = (r_i) \in R^\kappa$, there exists an $\mathcal{S}(\mathcal{T})$-measurable map $f_\mathbf{r} : X \to [0,1]^\kappa$ such that $\lambda_\mathbf{r} :$ Borel$([0,1]^\kappa) \to R$ given by $a \mapsto \eta(a \circ f_\mathbf{r})$ is an observable on R.*

Proof. It is an application of Corollary 2.9 and Definition 3.1. □

The spectral representation of a σ-complete Riesz MV-algebra allows us to obtain a more refined version of the integral representation of a state in the context of probability Riesz tribes. We recall that a probability Riesz tribe is a pair $(\mathcal{T}, s)$, where $\mathcal{T}$ is a Riesz tribe and s is a σ-additive state. We also note that for any κ-dimensional observable $\mathcal{X} : \mathrm{Borel}([0,1]^\kappa) \to \mathcal{T}$, the composition $s_\mathcal{X} = s \circ \mathcal{X}$ is a σ-additive state on $\mathrm{Borel}([0,1]^\kappa)$, while the function $\mu_{s_\mathcal{X}} : \mathcal{B}([0,1]^\kappa) \to [0,1]$, defined by $\mu_{s_\mathcal{X}}(E) = s_\mathcal{X}(\chi_E)$ is a probability measure on $\mathcal{B}([0,1]^\kappa)$, the σ-algebra of Borel subsets of $[0,1]^\kappa$.

Definition 3.4. For any κ-dimensional observable $\mathcal{X} : \mathrm{Borel}([0,1]^\kappa) \to (\mathcal{T}, s)$, the composition $s_\mathcal{X}$ will be called a *distribution state* of $\mathcal{X}$.

Proposition 3.5. *Let $(\mathcal{T}, s)$ be a probability Riesz tribe, $\{\lambda_f\}$ be its spectral representation, and $\mu_f : \mathcal{B}([0,1]) \to [0,1]$ be the measure associated to the distribution state of λ_f. Then*

$$s(f) = \int_0^1 y d\mu_f(y).$$

Proof. Recalling Theorem 1.2,

$$s(f) = \int_X f(x) d\mu_s,$$

where $\mu_s : \mathcal{S}(\mathcal{T}) \to [0,1]$ is a probability measure on the σ-algebra $\mathcal{S}(\mathcal{T}) = \{A \subseteq X \mid \chi_A \in \mathcal{T}\}$ given by $\mu_s(A) = s(\chi_A)$.

Since every function in $\mathcal{T}$ is $\mathcal{S}(\mathcal{T})$-measurable, for any Borel subset $E \in \mathcal{B}([0,1])$, $f^{-1}(E) \in \mathcal{S}(\mathcal{T})$. Therefore, any $f \in \mathcal{T}$ defines a pushforward measure $\mu_B : \mathcal{B}([0,1]) \to [0,1]$ as $\mu_B(E) = \mu_s(f^{-1}(E)) = s(\chi_{f^{-1}(E)})$. Since $\chi_{f^{-1}(E)} = \chi_E \circ f = \lambda_f(\chi_E)$, it follows that $s(\chi_{f^{-1}(E)}) = (s \circ \lambda_f)(\chi_E) = \mu_f(E)$, with μ_f being the probability measure associated to the state $s \circ \lambda_f : \mathrm{Borel}([0,1]) \to [0,1]$.

Applying now the well-known property of change of variables of pushforward measures (we are considering the space of integrable functions $L^1(\mathcal{B}([0,1]))$ endowed with the measure μ_B and the change of variables given by g being the identity function), for any $f \in \mathcal{T}$ we deduce the following, settling the claim.

$$s(f) = \int_X f(x) d\mu_s(x) = \int_0^1 y d\mu_B(y) = \int_0^1 y d\mu_f(y).$$

□

4. Conditional expectation operator

In this section we follow the general theory of Riesz spaces and define a *conditional observable* taking inspiration from the conditional expectation operator that can be found, for example, in [13, Section 233]. Once again, we

fundamentally use the fact that Borel$([0,1]^\kappa)$ is the κ-generated free algebra in $\mathbf{RMV}_\sigma$, avoiding the necessity of endowing an MV-algebra with a binary ring-like product, as was done e.g. in [10, 24].

Theorem 4.1 (Radon-Nikodym theorem of observables)**.** *Let $(\mathcal{T}, s)$ be a Riesz tribe with a σ-additive state s, let $\mathcal{R} \subseteq \mathcal{T}$ be a sub-Riesz tribe and $\mathcal{X} \in \mathcal{O}_\kappa(\mathcal{T})$. For any $a \in$ Borel$([0,1]^\kappa)$, there exists a (unique up to μ_s-null sets) function $\widetilde{a} \in \mathcal{R}$ such that*

$$\int_A \mathcal{X}(a) d\mu_s = \int_A \widetilde{a} d\mu_s$$

for any $A \in \mathcal{S}(\mathcal{R}) = \{B \subseteq X \mid \chi_B \in \mathcal{R}\}$. The measure μ_s is defined from $\mathcal{S}(\mathcal{T})$ to $[0,1]$ by $A \mapsto s(\chi_A)$.

Proof. Let us fix an arbitrary $a \in$ Borel$([0,1]^\kappa)$ and let ν be the measure on $\mathcal{S}(\mathcal{R})$ defined by $A \mapsto \int_A \mathcal{X}(a) d\mu_s$.

If $A \in \mathcal{S}(\mathcal{T})$ is a set such that $\mu_s(A) = 0$, then, [2, Theorem 6.2] implies that $\int_X \chi_A d\mu_s = 0 = \int_A d\mu_s$. By definition, for any $a \in$ Borel$([0,1]^\kappa)$ we have $\mathcal{X}(a) \leq \mathbf{1} \in \mathcal{R}$ and $\mu_s(A) = 0$, from which it follows that $\nu(A) = \int_A \mathcal{X}(a) d\mu_s = 0$, and ν is absolutely continuous with respect to μ_s.

Since $\mu_s(X) = s(\chi_X) = s(\mathbf{1}) = 1$ and $\nu(X) = \int_X \mathcal{X}(a) d\mu_s = s(\mathcal{X}(a)) \leq 1$, both measures are finite. Whence, the classical Radon-Nikodym Theorem implies, since $\mathcal{R}$ is exactly the tribe of all $\mathcal{S}(\mathcal{R})$-measurable functions, that there exists $\widetilde{a} \in \mathcal{R}$ such that, for any $A \in \mathcal{S}(\mathcal{R})$,

$$\int_A \mathcal{X}(a) d\mu_s = \int_A \widetilde{a} d\mu_s.$$

Moreover, $\widetilde{a}$ is unique up to a μ_s-null set, that is, if h satisfies the same property of $\widetilde{a}$, then $\mu_s(\{x \in X \mid h(x) \neq \widetilde{a}(x)\}) = 0$. □

In order to use Theorem 4.1 for obtaining a well-defined notion of conditional observable, we need to identify almost surely equal functions. We recall that $\mathcal{T}_s$ denotes the Riesz MV-algebra obtained from $\mathcal{T}$ identifying μ_s-equal everywhere functions with respect to the probability space $(X, \mathcal{S}(\mathcal{T}), \mu_s)$, in which countable suprema are defined pointwise, as proved in Lemma 2.10.

Definition 4.2. The *conditional expectation operator* of $\mathcal{X}$ with respect to $\mathcal{R}$ is the map $\mathcal{E}(\mathcal{X} \mid \mathcal{R}) : \text{Borel}([0,1]^\kappa) \to \mathcal{R}_s$ defined by $\mathcal{E}(\mathcal{X} \mid \mathcal{R})(a) = \tilde{a}$, as given in Theorem 4.1.

Thus, for any $A \in \mathcal{S}(\mathcal{R})$ and any $a \in$ Borel$([0,1]^\kappa)$,

$$\int_A \mathcal{X}(a) d\mu_s = \int_A \mathcal{E}(\mathcal{X} \mid \mathcal{R})(a) d\mu_s.$$

Lemma 4.3. *The above defined conditional expectation operator is an operator that preserves the top element, complements and scalar multiplication.*

Proof. Fixed $\mathcal{R} \subseteq \mathcal{T}$, we start by proving that $\mathcal{E}(\mathcal{X} \mid \mathcal{R})(\mathbf{1}) = \mathbf{1}$. Indeed, since $\mathbf{1}$ belongs to $\mathcal{R}$, it is already $\mathcal{S}(\mathcal{R})$-measurable and it follows that $\tilde{\mathbf{1}} = \mathbf{1}$ by the uniqueness of $\tilde{\mathbf{1}}$.

The additivity of the map is equivalent to asking for additivity when $\oplus$ coincides with the standard group operation $+$ between functions. Therefore, it follows directly from the linearity of the integral. The same happens for complements and scalar operation. Indeed, for any $A \in \mathcal{S}(\mathcal{R})$,

$$\begin{aligned}\int_A \mathcal{E}(\mathcal{X} \mid \mathcal{R})(a+b)d\mu_s &= \int_A \mathcal{X}(a+b)d\mu_s = \int_A (\mathcal{X}(a) + \mathcal{X}(b))\, d\mu_s \\ &= \int_A \mathcal{X}(a)d\mu_s + \int_A \mathcal{X}(b)d\mu_s \\ &= \int_A \mathcal{E}(\mathcal{X} \mid \mathcal{R})(a)d\mu_s + \int_A \mathcal{E}(\mathcal{X} \mid \mathcal{R})(b)d\mu_s \\ &= \int_A (\mathcal{E}(\mathcal{X} \mid \mathcal{R})(a) + \mathcal{E}(\mathcal{X} \mid \mathcal{R})(b))\, d\mu_s,\end{aligned}$$

from which we deduce $\mathcal{E}(\mathcal{X} \mid \mathcal{R})(a+b) = \mathcal{E}(\mathcal{X} \mid \mathcal{R})(a) + \mathcal{E}(\mathcal{X} \mid \mathcal{R})(b)$, since Theorem 4.1 entails the uniqueness of $\mathcal{E}(\mathcal{X} \mid \mathcal{R})(a+b)$ up to μ_s-null sets. Similarly,

$$\begin{aligned}\int_A \mathcal{E}(\mathcal{X} \mid \mathcal{R})(a^*)d\mu_s &= \int_A \mathcal{X}(a^*)d\mu_s = \int_A (\mathbf{1} - \mathcal{X}(a))d\mu_s \\ &= \int_A \mathbf{1}d\mu_s - \int_A \mathcal{E}(\mathcal{X} \mid \mathcal{R})(a)d\mu_s \\ &= \int_A (\mathbf{1} - \mathcal{E}(\mathcal{X} \mid \mathcal{R})(a))\, d\mu_s = \int_A (\mathcal{E}(\mathcal{X} \mid \mathcal{R})(a))^*\, d\mu_s.\end{aligned}$$

Finally, for any $\alpha \in [0,1]$,

$$\begin{aligned}\int_A \mathcal{E}(\mathcal{X} \mid \mathcal{R})(\alpha a)d\mu_s &= \int_A \mathcal{X}(\alpha a)d\mu_s = \int_A \alpha\mathcal{X}(a)d\mu_s \\ &= \int_A \alpha\mathcal{E}(\mathcal{X} \mid \mathcal{R})(a)d\mu_s.\end{aligned}$$

□

Corollary 4.4. *The composition $s \circ \mathcal{E}(\mathcal{X} \mid \mathcal{R})\colon$ Borel$([0,1]^\kappa) \to [0,1]$ is a well-defined state and it coincide with the distribution state of $\mathcal{X}$.*

Proof. For any $a \in \mathrm{Borel}([0,1]^\kappa)$, $s(\mathcal{E}(\mathcal{X} \mid \mathcal{R})(a)) = \int_X \mathcal{E}(\mathcal{X} \mid \mathcal{R})(a)d\mu_s$, which is well-defined by Theorem 4.1. Moreover, the same result implies that $s(\mathcal{E}(\mathcal{X} \mid \mathcal{R})(a)) = s_{\mathcal{X}}(a)$ for any $a \in \mathrm{Borel}([0,1]^\kappa)$. □

Finally, to let $\mathcal{X}^{\mathcal{R}} : \mathrm{Borel}([0,1]^\kappa) \to \mathcal{R}_s$ to be the unique extension of the assignment $\pi_i \mapsto \widetilde{\pi}_i$, given in Theorem 4.1. Inquiring how $\mathcal{X}$, $\mathcal{E}(\mathcal{X} \mid \mathcal{R})$ and $\mathcal{X}^{\mathcal{R}}$ are related on a generic function of $\mathrm{Borel}([0,1]^\kappa)$, we get the following result.

Proposition 4.5. *If $\mathcal{X}(\pi_i) \in \mathcal{R}$, for $i = 1, \ldots, n$. Then $\mathcal{E}(\mathcal{X} \mid \mathcal{R})$ is an observable and it coincide with $\mathcal{X}^{\mathcal{R}}$. Thus, $\mathcal{X}^{\mathcal{R}} = [\cdot]_s \circ \mathcal{X}$ and*

$$\int_A \mathcal{X}(a) d\mu_s = \int_A \mathcal{X}^{\mathcal{R}}(a) d\mu_s. \tag{4}$$

Proof. By hypothesis, $Im(\mathcal{X}) \subseteq \mathcal{R}$. Whence, each function in $Im(\mathcal{X})$ is already $\mathcal{S}(\mathcal{R})$-measurable and the uniqueness given by Theorem 4.1 implies that for any $a \in \text{Borel}([0,1]^\kappa)$, $\mathcal{E}(\mathcal{X} \mid \mathcal{R})(a) = [\mathcal{X}(a)]_s$, that it, they are almost everywhere equal. By Lemma 2.10 the composition $[\cdot]_s \circ \mathcal{X}$ is a well defined observable on $\mathcal{R}_s$ and since $\mathcal{E}(\mathcal{X} \mid \mathcal{R})(\pi_i) = [\mathcal{X}(\pi_i)]_s = \tilde{\pi}_i = \mathcal{X}^{\mathcal{R}}(\pi_i)$, for any $i \in \kappa$, the claim is settled. □

We remark that the hypothesis of $\mathcal{X}(\pi_i) \in \mathcal{R}$ is crucial for Proposition 4.5, as shown in the next example, for which we thank the anonymous referee.

Example 4.6. Consider the observable $\mathcal{X} : \text{Borel}([0,1]) \to \text{Borel}([0,1])$ defined by $\mathcal{X}(a) = a$ and endow it with the state obtained by integrating according to the Lebesgue measure on $\mathcal{B}([0,1])$. Take $\mathcal{R} \subseteq \text{Borel}([0,1])$ to be the Riesz tribe consisting of all constant functions. In this case $f = id = \tilde{f}$ and

$$\int_{[0,1]} f d\mu_s = \int_{[0,1]} \tilde{f} d\mu_s.$$

Now, for $a = \chi_E$ with $E = \left[\frac{3}{4}, 1\right]$, it is easy to prove that Equation (4) does not hold, so contradicting the claim. Indeed

$$\int_{[0,1]} a \circ f d\mu_s = \int_{[0,1]} \chi_E(x) d\mu_s = \frac{1}{4},$$

while

$$\int_{[0,1]} a \circ \tilde{f} d\mu_s = \int_{[0,1]} \chi_E\left(\frac{\mathbf{1}}{\mathbf{2}}(x)\right) d\mu_s = 0.$$

From a different perspective, we can define the expectation (or mean value) of an observable, rather than the conditional expectation observable as follows. Let $(\mathcal{T}, s)$ be a probability Riesz tribe, and let μ_s be the corresponding σ-additive measure on $\mathcal{S}(\mathcal{T})$. If $\mathcal{X}$ is a one-dimensional observable, by Theorem 2.3 there exists an $\mathcal{S}(\mathcal{T})$-measurable function f such that $\mathcal{X}(a) = a \circ f$, $a \in \text{Borel}([0,1])$, then we define the *expectation*, $Exp_\mu(\mathcal{X})$, of $\mathcal{X}$ in the state s by

$$Exp_\mu(\mathcal{X}) = \int_X f(x)\, \mathrm{d}\mu_s(x).$$

Thus, in the one-dimensional case the expectation of $\mathcal{X}$ is the *classical* expected value of the random variable associated to $\mathcal{X}$.

Now, let $\mathcal{X}$ be a κ-dimensional observable. If f_i denotes $\mathcal{X}(\pi_i)$, for each $i \in \kappa$, we define a one-dimensional observable $\mathcal{X}_i(a) = a \circ f_i$, $a \in \text{Borel}([0,1])$, and for

each $\mathcal{X}_i$ we define an expectation $Exp_\mu(\mathcal{X}_i)$ in the state s. Then the expectation of $\mathcal{X}$ in the state s is a vector

$$Exp_\mu(\mathcal{X}) = (Exp_\mu(\mathcal{X}_i)_{i\in\kappa}).$$

5. Stochastic processes on Riesz tribes

Finally, we are ready to define stochastic processes.

Definition 5.1. A (κ-dimensional) *stochastic process* on the Riesz tribe $\mathcal{T} \subseteq [0,1]^X$ is a sequence of (κ-dimensional) observables.

Thus, in our setting a stochastic process is a sequence of homomorphisms in the infinitary variety $\mathbf{RMV}_\sigma$. Since we work with countably many observables, we can obtain many classical results and notions in a purely algebraic way. One example is the following results.

Proposition 5.2. *Let $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ be a κ-dimensional stochastic process posed in $\mathcal{T}$. The map $\mathcal{X}^\uparrow : Borel([0,1]^\kappa) \to \mathcal{T}$ and $\mathcal{X}^\downarrow : Borel([0,1]^\kappa) \to \mathcal{T}$ induced respectively by the assignments $\pi_i \mapsto \bigvee_n \mathcal{X}_n(\pi_i)$ and $\pi_i \mapsto \bigwedge_n \mathcal{X}_n(\pi_i)$ are well-defined observables.*

Given a stochastic process $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$, we shall call $\mathcal{X}^\uparrow$ and $\mathcal{X}^\downarrow$ the upper and lower limit observable, respectively.

In the same spirit, the very definition of a process allows us to define the *evolution* of a one-dimensional process by considering its joint observable, $\mathcal{J}_\omega$, defined as the joint of all $\mathcal{X}_n$. Moreover, we can talk about its distribution state, s_ω, defined as $s \circ \mathcal{J}_\omega$. Notice that, in classical probability, the joint distribution of countably many random variables $\{f_n\}_{n\in\mathbb{N}}$ is usually defined by considering finite subsets of the f_n's. In our case the outcome is the same, as justified by the following lemma.

Lemma 5.3. *Let $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ be a stochastic process posed in $\mathcal{T} \subseteq [0,1]^X$, and let $a \in Borel([0,1]^\omega)$ be a function that depends only of a finite number k of variables $\{x_{i_1}, \dots, x_{i_k}\}$. Then $\mathcal{J}_\omega(a) = \mathcal{J}_k(a)$, where $\mathcal{J}_k$ is the joint observable of $\mathcal{X}_{i_1}, \dots, \mathcal{X}_{i_k}$.*

Proof. Let $f = (f_n)$ be the function such that $\mathcal{J}_\omega(a) = a \circ f$. Notice that for any $b \in \mathrm{Borel}([0,1]^k)$, $\mathcal{J}_k(b) = b \circ (f_{i_1}, \dots, f_{i_k})$. Whence, for any $x \in X$

$$\begin{aligned}\mathcal{J}_\omega(a)(x) =& a(x_{i_1}, \dots, x_{i_k})(f_n(x))_n = \\ =& a(f_{i_1}(x), \dots, f_{i_k}(x)) = a(f_{i_1}(x), \dots, f_{i_k}(x)) = \mathcal{J}_k(a)(x).\end{aligned}$$

□

Definition 5.4. Let $(\mathcal{T}, s)$ be a probability Riesz tribe and let $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ be an increasing sequence of subtribes of $\mathcal{T}$. We will say that a κ-dimensional stochastic process $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ on $\mathcal{T}$ is *adapted* to $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ if, for any $n \in \mathbb{N}$ and any $a \in \mathrm{Borel}([0,1]^\kappa)$, $\mathcal{X}_n(a)$ is $\mathcal{S}(\mathcal{T}_n)$-measurable. Such a sequence of subtribes will be called a *filtration*.

Given any one dimensional process $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ it is possible to build a natural sequence of subtribes that make the process adapted to them.

Proposition 5.5. *Let $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ be a one dimensional process. Then, denoted by $\mathcal{J}_m$ the joint observable of $\mathcal{X}_1,\dots,\mathcal{X}_m$, the subtribes $Im(\mathcal{J}_m)$ are a filtration for the process. Moreover, for any $a \in Borel([0,1])$ and any $i = 1,\dots,m$ the function $\mathcal{X}_i(a)$ is $\mathcal{S}(Im(\mathcal{J}_m))$-measurable.*

Proof. It follows from Proposition 2.2 and [2] that $Im(\mathcal{X}_m)$ is the Riesz tribe of all $\mathcal{S}(Im(\mathcal{X}_m))$-measurable functions. Whence, $\mathcal{X}_m(a)$ is $\mathcal{S}(Im(\mathcal{X}_m))$-measurable, for any m and any $a \in \mathrm{Borel}([0,1])$. With the same argument, $\mathcal{J}_m(a)$ is $\mathcal{S}(Im(\mathcal{J}_m))$-measurable, for any $a \in \mathrm{Borel}([0,1]^k)$.

The sequence $\{Im(\mathcal{J}_m))\}_{m\in\mathbb{N}}$ is indeed increasing, since $\mathrm{Borel}([0,1]^m) \hookrightarrow \mathrm{Borel}([0,1]^{m+1})$ implies that $Im(\mathcal{J}_m) \subseteq Im(\mathcal{J}_{m+1})$.

Finally, take $i = 1,\dots,m$. In order to settle the claim, it is enough to prove that $Im(\mathcal{X}_i) \subseteq Im(\mathcal{J}_m)$. By Theorem 2.3, $\mathcal{X}_i(a) = a \circ f_i$, with $f_i = \mathcal{X}_i(id)$ and $a \in \mathrm{Borel}([0,1])$, while $\mathcal{J}_m(b) = b \circ f$, with $f = (f_1,\dots,f_m)$ and $b \in \mathrm{Borel}([0,1]^m)$. Take $a\circ f_i \in Im(\mathcal{X}_i)$. It follows that $a\circ f_i = a\circ(\pi_i\circ f) = (a\circ\pi_i)\circ f$. Since $a \circ \pi_i \in \mathrm{Borel}([0,1]^m)$, $(a \circ \pi_i) \circ f = \mathcal{J}_k(a \circ \pi_i) \in Im(\mathcal{J}_m)$, settling the claim. □

The filtration $Im(\mathcal{J}_k)$ is actually build starting from the images of the observables we are joining. Indeed, the following holds.

Proposition 5.6. *Let $\mathcal{J}_\kappa$ be the joint observable of the one-dimensional observables $\mathcal{X}_{i_1},\dots,\mathcal{X}_{i_\kappa}$, with $\kappa \le \omega$. Then $Im(\mathcal{J}_\kappa)$ is the Riesz tribe generated by $\mathcal{X}_{i_1}(id),\dots,\mathcal{X}_{i_\kappa}(id)$. Moreover, if $E = \Pi_{j=1}^{\kappa} E_j$, with $E_j \in \mathcal{B}([0,1])$, $\mathcal{J}_k(\chi_E) = \bigodot_{j\in\kappa} \mathcal{X}_{i_j}(\chi_{E_j}) = \bigwedge_{j\in\kappa} \mathcal{X}_{i_j}(\chi_{E_j})$.*

Proof. We first remark that $Im(\mathcal{X}_{i_j}) \subseteq Im(\mathcal{J}_n)$ for any $j = 1,\dots,n$, as proved in Proposition 5.5. Thus, denote by $\mathcal{T}_\kappa$ the tribe generate by $\mathcal{X}_{i_1}(id),\dots,\mathcal{X}_{i_\kappa}(id)$ in $\mathcal{T}$, it is trivial to notice that $\mathcal{T}_\kappa \subseteq Im(\mathcal{J}_\kappa)$.

To prove the converse inclusion, take $h \in Im(\mathcal{J}_\kappa)$. Then, for some $a \in \mathrm{Borel}([0,1]^\kappa)$, we have $h(x) = a((f_{i_j}(x))_{j\in\kappa})$, with $f_{i_j} = \mathcal{X}_{i_j}(id) \in Im(\mathcal{X}_{i_j}) \subseteq Im(\mathcal{J}_\kappa)$. Notice that $a \in \mathrm{Borel}([0,1]^\kappa)$ is, in fact, an $\mathbf{RMV}_\sigma$–combination of the projections π_j, with $j \in \kappa$. In other words, there exists a term τ such that $a = \tau((\pi_j)_{j\in\kappa})$. Consider now the homomorphism $\eta : \mathrm{Borel}([0,1]^\kappa) \to \mathcal{T}_\kappa$ obtained by extending the assignment $\pi_j \mapsto f_{i_j}$, with $j \in \kappa$. Since all operations are defined pointwise and η is a homomorphism,

$$h = a \circ ((f_{i_j})_{j\in\kappa}) = \tau((\eta(\pi_j)_{j\in\kappa}) = \eta(\tau(\pi_j)_{j\in\kappa}) = \eta(a) \in \mathcal{T}_\kappa,$$

which settles the first part of the claim.

Finally, $\mathcal{X}_{i_j}(\chi_{E_j}) = \chi_E \circ f_j = \chi_{f_j^{-1}(E_j)}$, where the latter equality is easily deduced by direct computation, indeed $(\chi_{E_j} \circ f_j)(x) = 1$ if, and only if, $f_j(x) \in E_j$ and therefore if, and only if, $x \in f_j^{-1}(E_j)$, that is if, and only if $\chi_{f_j^{-1}(E_j)} = 1$.

Analogously, if $f = (f_j)_{j\in\kappa}$ it holds that $\mathcal{J}_k(\chi_E) = \chi_E \circ f = \chi_{f^{-1}(E)}$ and it is easily seen, given the definition of f, that $f^{-1}(E) = \bigcap_{j\in\kappa} f_j^{-1}(E_j)$. Thus,

$$\begin{aligned}
\chi_{f^{-1}(E)}(x) = 1 &\Leftrightarrow \chi_{\bigcap_{j\in\kappa} f_j^{-1}(E_j)}(x) = 1 \\
&\Leftrightarrow \left(\bigwedge_{j\in\kappa} \chi_{f_j^{-1}(E_j)}\right)(x) = 1 \\
&\Leftrightarrow \bigwedge_{j\in\kappa} \chi_{f_j^{-1}(E_j)}(x) = 1 \\
&\Leftrightarrow \bigodot_{j\in\kappa} \chi_{f_j^{-1}(E_j)}(x) = 1,
\end{aligned}$$

settling the claim. □

Example 5.7. Consider the sequence $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ in which each $\mathcal{X}_n$ coincides with the observable given in Example 2.4. In this case, $\mathcal{X}^{\uparrow} = \mathcal{X}^{\downarrow} = \mathcal{X}$ and the process is naturally adapted to the constant filtration $Im(\mathcal{X})$.

Moreover, for any choice of indexes $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$, the joint observable $\mathcal{J}_m^i$ of $\mathcal{X}_{i_1}, \ldots, \mathcal{X}_{i_m}$ and $\mathcal{J}_m^j$ of $\mathcal{X}_{j_1}, \ldots, \mathcal{X}_{j_m}$ coincide. If $\mathcal{X}(a) = a \circ f$, then $\mathcal{J}_m^i(a) = \mathcal{J}_m^j(a) = a \circ (f, \ldots, f)$, where f appears m times.

Definition 5.8. Let $(\mathcal{T}, s)$ be a Riesz probability tribe and let $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ be a sequence of observables on $\mathcal{T}$. The sequence is called *independent* if for any $n \in \mathbb{N}$ and any $i_1, \ldots, i_n$ there exists an n-ary operator

$$\beta : Im(\mathcal{X}_{i_1}) \times \cdots \times Im(\mathcal{X}_{i_n}) \to Im(\mathcal{J}_n)$$

such that for all $a_1, \ldots, a_n \in \mathrm{Borel}([0,1])$ we have

$$s(\beta(\mathcal{X}_{i_1}(a_1), \ldots, \mathcal{X}_{i_n}(a_n))) = s(\mathcal{X}_{i_1}(a_1)) \cdot \ldots \cdot s(\mathcal{X}_{i_n}(a_n)).$$

Defining a suitable notion of independence for probability MV-algebras was an open problem left in [24]. This problem was tackled and solved in [18]. Thus, we remark that the definition of independent observables, rather than independent algebras, is a generalization of [18, Definition 3.1] to the case of the probability tribes $(Im(\mathcal{X}_{i_1}), s), \ldots, (Im(\mathcal{X}_{i_n}), s)$, which are required to be $(Im(\mathcal{J}_n), s, \beta)$-independent, where β is an n-linear map, rather than a bilinear one. Moreover, the codomain of β is, in fact, the tribe generated by the factors that appear in the domain.

As standard in probability theory, see [13, Definition 272A], sequence of σ-algebras $\mathcal{S}_n \subseteq \Sigma$ in the probability space (X, Σ, μ) is called independent if for $k \in \mathbb{N}$ and any $A_j \in \mathcal{S}_{i_j}$, $j = 1, \ldots, k$, $\mu(A_1 \cap \cdots \cap A_k) = \mu(A_1) \cdot \ldots \cdot \mu(A_k)$.

Proposition 5.9. *Let $\{\mathcal{X}_n\}_{k\in\mathbb{N}}$ be the process defined by $\mathcal{X}_n(a) = a \circ f_n$. If the sequence $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ is independent, with β being the usual product of functions, then the sequence $\{f_n\}_{n\in\mathbb{N}}$ is independent.*

Proof. Following [13, Proposition 272D], the sequence $\{f_n\}_{n\in\mathbb{N}}$ of classical random variables is independent if the σ-algebras $f_n^{-1}(\mathcal{B}([0,1]))$ are independent.

Let us first notice that, in our hypothesis, the σ-algebras $\mathcal{S}(Im(\mathcal{X}_n))$ are independent. Indeed, for any $A_j \in \mathcal{S}(\mathcal{X}_{i_j})$, $j = 1, \ldots, k$, we have $\chi_{A_j} \in Im(\mathcal{X}_{i_j})$, whence $s(\beta(\chi_{A_1}, \cdots, \chi_{A_n})) = s(\chi_{A_1} \wedge \cdots \wedge \chi_{A_n}) = s(\chi_{A_1}) \cdot \ldots \cdot s(\chi_{A_n})$, that is, $\mu_s(A_1 \cap \cdots \cap A_k) = \mu_s(A_1) \cdot \ldots \cdot \mu_s(A_k)$.

Therefore, we shall prove that, for any $n \in \mathbb{N}$, $\mathcal{S}(Im(\mathcal{X}_n)) = f_n^{-1}(\mathcal{B}([0,1]))$.

Let $A \in \mathcal{S}(Im(\mathcal{X}_n))$. Then $\chi_A \in Im(\mathcal{X}_n)$ and there exists $a \in \mathrm{Borel}([0,1])$ such that $\chi_A = a \circ f_n$. Thus, $A = \chi_A^{-1}(\{1\})$, and therefore if $A = (a \circ f_n)^{-1}(\{1\})$, which is an element of ${}_n f^{-1}(\mathcal{B}([0,1]))$ since $a^{-1}(\{1\})$ is an element of $\mathcal{B}([0,1])$.

Conversely, let $A \in f_n^{-1}(\mathcal{B}([0,1]))$. There exists $E \in \mathcal{B}([0,1])$ such that $A = f_n^{-1}(E)$. We have that $x \in f_n^{-1}(E)$ if, and only if, $f_n(x) \in E$, which holds true if, and only if, $\chi_E(f_n(x)) = 1$, that is $\mathcal{X}_n(\chi_E) = 1$. Whence, $\chi_A = \mathcal{X}_n(\chi_E)$, and $\chi_A \in Im(\mathcal{X}_n)$, settling the claim. □

Conclusions

In this work we aimed at laying the ground for the investigation of probability theory within logic. We defined random variables and stochastic processes in a way that is entirely codified inside a logical system and we discussed some classical notions, as well as notions that already appear in non-classical logic, in order to test the effectiveness of our definition. We believe that this work is only a starting point for a deeper point of view of probability theory and that much more work can be done on this subject.

With an eye to a metamathematics of probability, we mention [1], where the author engaged in an analysis of theories of $[0,1]$-valued random variables. The main aim there is to provide an axiomatization for the theory whose models are spaces of equivalence classes of integrable functions that take value in a general enough metric space. The first part of [1] is focused on $[0,1]$-valued random variables and in this setting the author adds a binary connective that models that expectation of the absolute value of the distance of two random variables. The work builds on the author's notion of continuous logic, which is easily interpreted in the logic of Riesz MV-algebras. Thus, an investigation of the interactions with our framework is in order.

Finally, we close this paper by recalling two approaches to stochastic processes using techniques and notions of non-classical logic. This is not meant to be an exhaustive list of references, but an attempt to mention the approaches that, to the best of our knowledge, have more similarities (either in spirit or for the context) to our work.

In [14], the authors define Markov processes in an abstract sense, obtaining a modal logic in which the modality is interpreted in an appropriate conditional expectation operator. It would be interesting to have a better understanding of the interplay their notion of Markov processes with our definition of stochastic process.

In the PhD thesis [15] (and the references within), author and co-authors analyze stochastic processes in Riesz spaces from what seems to be a completely

different angle. Their work is based on the idea to work with operators in Riesz spaces and describe the main notions involved in stochastic processes from an abstract and point-free point of of view.

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