Dualities and algebraic geometry of Baire functions in Non-classical Logic

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Abstract

In this paper we aim at completing the study of σ-complete Riesz MV-algebras that started in [7]. To do so, we discuss polynomials, algebraic geometry and dualities in the infinitary variety of such algebras. In particular, we characterize the free objects as algebras of Baire-measurable functions and we generalize two dualities, namely the Marra-Spada duality and the Gelfand duality, obtaining a duality with basically disconnected compact Hausdorff spaces and an equivalence with Rickart C^*-algebras.
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1 Introduction

One of the most investigated many-valued logics is certainly Łukasiewicz logic, the main reason being the fact that its algebraic semantics has been proved to be a very malleable class of algebras, that carries many underlying relations with other areas of mathematics or theoretical computer science. One can see the appendixes of [22] to grasp the broad collection of tools and techniques from different fields that one can use to carry out state-of-art research on the topic.

A major contribution to the diffusion of MV-algebras is undoubtedly their celebrated categorical equivalence with lattice-ordered groups with a strong unit, D. Mundici's equivalence, that dates back to 1986. This result led the way for an investigation of MV-algebras inspired by the theory of groups, and one of the outcomes of this point of view has been the definition of the class of *Riesz MV-algebras*. These algebras stand to MV-algebras like vector spaces over $\mathbb{R}$ stand to groups. We refer to Section 2.1 and [8, 6, 22] for the missing preliminary notions.

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Riesz MV-algebras have proven to be a quite interesting and useful class of algebras in themselves. They provide the semantics for a conservative extension of Łukasiewicz logic, but more importantly they are successful in overcoming some of the limitations of MV-algebras. Without entering into the details, it is worth to stress that the presence of the scalar multiplication allows one to cross the bridge towards different areas of mathematics with a luggage of techniques that cannot be used in the setting of MV-algebras alone. Just to give some examples, we mention three references where these structures have been instrumental for the application of non-classical logic to other areas of research: In [5] term-functions in the language of Riesz MV-algebras are proved equivalent to some special type of artificial neural network; in [13] the authors give a different proof of a crucial result of subjective decision theory using Riesz MV-algebras, thus establishing a seminal link between these two areas of research; in [4] Riesz MV-algebras are deployed to define a generalized notion of random variable and stochastic process.

In this paper we will not deal with the totality of Riesz MV-algebras. Indeed, our starting point will be the class of algebras considered in [4]: the infinitary variety of σ-complete Riesz MV-algebras, denoted by $\mathbf{RMV}_\sigma$. These algebras were at the core of [7], where the authors define an infinitary propositional extension of Łukasiewicz logic, whose algebraic semantics is exactly $\mathbf{RMV}_\sigma$.

Another important point noted in [7] is the fact that the algebras in $\mathbf{RMV}_\sigma$ have a quite concrete representation, since they can be thought of as intervals of some Banach lattices. A Banach lattice is a lattice-ordered vector space over $\mathbb{R}$ which is norm-complete. In the special case of a vector space with a strong order unit, one can consider the class of those Banach lattices that are complete with respect to a norm that is induced by the strong unit. These spaces turned out to be all $\mathbb{R}$-valued algebras of continuous functions over compact and Hausdorff topological spaces. Any Dedekind σ-complete lattice-ordered vector space with a strong order unit is norm-complete, and therefore, when in addition one requires suitable topological properties, these algebras of functions turn out to be equivalent "à la Mundici" to σ-complete Riesz MV-algebras.

Moreover, in [7] σ-complete Riesz MV-algebras are proven to be an infinitary variety in which the free object over a finite number n of generators was characterized as the algebra of $[0, 1]$-valued and Borel measurable functions over $[0, 1]^n$, making more clear how these algebras can be linked to real-world problems and, in particular, to probability. Indeed, using σ-complete Riesz MV-algebras, in [4] the authors provide a generalized, algebraic version of the notion of random variable in a way that actually extends the classical and measure-theoretical notion in a quite natural fashion.

In this paper we aim at completing the algebraic investigation of σ-complete Riesz MV-algebras that started in [7], with an eye to algebraic geometry and polynomials. To fulfill this goal, after all needed preliminary notions, we dive into characterizing the free objects of $\mathbf{RMV}_\sigma$ in the case of an arbitrary set of generators. This is the content of Section 3, where we prove that the free X-generated algebras are the algebras of all $[0, 1]$-valued and Baire-measurable functions defined over $[0, 1]^X$, where X is an arbitrary set. In Section 4 we describe the Zariski topology given by the zerosets of term-functions and, starting from a general adjunction given in [3], we obtain dualities between subcategories of $\mathbf{RMV}_\sigma$ and Baire subsets of hypercubes. In Section 5 we restrict the well-known Gelfand-Naimark-Yosida-Krein-Kakutani duality to the case of σ-complete objects, obtaining a duality with a suitable category of topological spaces, as well as an equivalence with a subcategory of unital and commutative C^*-algebras, namely the Rickart C^*-algebras, and an equivalence with σ-complete Boolean algebras. In Section 6 we discuss the notion of polynomial completeness given in [1]: loosely speaking, the results of this section allow us to think of term functions in Riesz MV-algebras as polynomials in a more classical algebraic sense.

Finally, we would like to remark that the results of this paper provide a step forward in the study of Riesz MV-algebra, but they also serve a different purpose. Indeed, as mentioned already, σ-complete Riesz MV-algebras have been instrumental in the definition of a well-behaved generalization of the notion of random variable in [4]: the algebras $A \in \mathbf{RMV}_\sigma$ play the role of σ-algebras of events, while morphisms from free objects to A play the role of generalized random variables. Whence, we believe that the investigation we carried out in this paper can lead the way to a fruitful merging of non-classical logic and algebraic statistics.

Indeed, in [15], the authors define algebraic statistics as a discipline *concerned with statistical models that can be described, in some way, via polynomials [...] Innovations have entered from the use of the apparatus of polynomial rings: algebraic varieties, ideals, elimination, quotient operations and so on.* Algebraic statistics is a relatively new area of research that aims at using more and more sophisticated tools from algebra in order to deal with important aspects of statistics such as the modeling of an experiment. Moving in this direction, in [24] the authors outline a path towards the ambitious goal of rewriting the foundations of probability and statistics via algebraic geometry. Moreover, Chapter 4 of [24] is devoted at making a connection between design (of experiments) and Boolean algebras: Boolean polynomials are deployed to represent different designs.

Thus, building on the idea that Baire functions are polynomials, on the relation between C^*-algebras and quantum logic, and on the fact that term functions of $IRL(X)$ behave like polynomials in a more classical sense, this paper can be seen as a step towards providing the tools of algebraic geometry needed to investigate the designs of an experiment with the point of view of algebraic statistics and techniques and formalism of non-classical logic.

2 Preliminaries and notations

2.1 Algebraic notions

Our starting point is an infinitary conservative extension of Łukasiewicz logic, namely the Infinitary Riesz Logic defined in [7] and denoted by $\mathcal{IRL}$. This logic is obtained by combining the propositional variables with the Lukasiewicz operators $0, \oplus, \neg$, the Riesz operators ∇_α ($\alpha \in [0,1]$) and $\bigvee$, a countable operator that models a countable disjunction. An additional connective $\odot$ can be defined as $\varphi \odot \psi := \neg(\neg\varphi \oplus \neg\psi)$. Note that a formula in $\mathcal{IRL}$ may be infinite, but it always has countable length.

For sake of completeness, let us recall the axioms of $\mathcal{IRL}$.

$(L1)$ $\varphi \to (\psi \to \varphi)$

$(L2)$ $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

$(L3)$ $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

$(L4)$ $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

$(R1)$ $\nabla_\alpha(\varphi \to \psi) \leftrightarrow (\nabla_\alpha\varphi \to \nabla_\alpha\psi)$

$(R2)$ $\nabla_{(\alpha\odot\neg\beta)}\varphi \leftrightarrow (\nabla_\beta\varphi \to \nabla_\alpha\varphi)$

$(R3)$ $\nabla_\alpha(\nabla_\beta\varphi) \leftrightarrow \nabla_{\alpha\cdot\beta}\varphi$

$(R4)$ $\nabla_1\varphi \leftrightarrow \varphi$

$(S1)$ $\varphi_k \to \bigvee_{n\in\mathbb{N}} \varphi_n$ for any $k \in \mathbb{N}$.

Note that (L1)-(L4) are the axioms of Łukasiewicz logic, (R1)-(R4) are the additional axioms of the logic of Riesz MV-algebras, the deduction rules will be the *Modus Ponens* and the following:

$$\text{(SUP)} \qquad \frac{(\varphi_1 \to \psi), \ldots, (\varphi_k \to \psi) \ldots}{\bigvee_{n\in\mathbb{N}} \varphi_n \to \psi}$$

Models of $\mathcal{IRL}$ turned out to be σ-complete Riesz MV-algebras, that is, algebras of type $(A, \oplus, \neg, \{\alpha\}_{\alpha\in[0,1]}, 0)$ such that $(A, \oplus, 0)$ is an Abelian monoid, $\neg$ is an involution, the operators $\{\alpha\}_{\alpha\in[0,1]}$ model a scalar multiplication, the relation $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ always holds, and A is closed under both countable suprema and infima.

These algebras are unit intervals of Dedekind σ-complete *Riesz spaces* with a strong unit, that is, Dedekind σ-complete vector lattices with a strong order unit, where further details on Riesz Spaces can be found e.g. in [19]. In more detail, the functor that gives the equivalence is the so-called Mundici's functor Γ. On objects, it is easily described as follows: for any vector lattice with a distinguished strong order unit (V, u), $\Gamma(V, u)$ is the *unit interval* $[0, u]_V = \{x \in V \mid 0 \leq x \leq u\}$. The scalar multiplication on $[0, u]_V$ is defined as in (V, u), while the other operations are given by $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$.

To give a more concrete intuition of σ-complete Riesz MV-algebras, we recall that from [7, Theorem 4.6] any such algebra is isomorphic to an algebra $C(X) = \{f : X \to [0,1] \mid f \text{ is continuous}\}$ for some compact, Hausdorff and basically disconnected space X, see Section 2.2 for the topology-related definitions and further details on these facts. Whence, every σ-complete Riesz MV-algebra is semisimple and it is an algebra of continuous functions.

If we denote by $\mathbf{RMV}_\sigma$ the class of σ-complete Riesz MV-algebras, it follows from [7, Theorem 5.3] that $\mathbf{RMV}_\sigma$ is the variety of infinitary algebras of type

$$0, \oplus, \neg, \{\alpha\}_{\alpha\in[0,1]}, \bigvee$$

which verify all polynomial identities that hold true in $[0,1]$ equipped with the following operations: $x \oplus y = \min(x + y, 1)$, $\neg x = 1 - x$, $\alpha(x) = \alpha x$, and

$\bigvee_n a_n = \sup_n a_n$. As a consequence (see e.g. [27]), the free object in $\mathbf{RMV}_\sigma$, over a set X, does exist and coincides with the algebra of all term functions from $[0,1]^X$ to $[0,1]$, that in [7] was denoted by $\mathcal{RT}_n$ when $|X| = n$. Moreover, from [7, Proposition 3.12] we get that the Lindenbaum -Tarski algebra of $\mathcal{IRL}$ is also isomorphic to the free ω-generated algebra in $\mathbf{RMV}_\sigma$. Whence, any formula in the language of $\mathcal{IRL}$ can be seen as a term-function.

To give more uniform notations, in this work $IRL(X)$ will denote all of these isomorphic algebras. That is, $IRL(X)$ is the algebra of term functions $f : [0,1]^X \to [0,1]$ in the language of σ-complete Riesz MV-algebras, it is the free X-generated algebra, and when $|X| \leq \omega$, $IRL(X)$ will also denote the Lindenbaum-Tarski algebra of the logic $\mathcal{IRL}$ built upon the set X viewed as a set of propositional variables. The elements of $IRL(X)$ will be called *IRL-polynomials.*

We note that in [7, Theorem 5.6] it is proved that, for a finite set X, $IRL(X)$ is the free algebra generated by the projection functions. It is easy to check that the argument used there is quite standard and it carries over to sets of arbitrary cardinality.

Notation. For any $A \in \mathbf{RMV}_\sigma$, we shall use

- $Id(A)$ to denote the MV-ideals of A, and we shall refer to them as *ideals*;
- $Max(A)$ to denote the maximal members of $Id(A)$, henceforth *maximal ideals*;
- $Id_\sigma(A)$ the MV-ideals of A that are closed under countable suprema (therefore, in correspondence with congruences in A), referred to as *σ-ideals*;
- $Max_\sigma(A)$ to denote the maximal members in $Id_\sigma(A)$, referred to as *maximal σ-ideals.*

Note that $Max(A) \cap Id_\sigma(A) \subseteq Max_\sigma(A)$ and the inclusion can be strict, as it is not guaranteed that an element of $Max_\sigma(A)$ is maximal for the MV-reduct. Elements of $Max(A) \cap Id_\sigma(A)$ will be explicitly referred to as *MV-maximal σ-ideals.*

We also recall that congruences in Riesz MV-algebras correspond to MV-ideals, see [8, Remarks 2 and 3].

2.2 Topological notions and lemmas on algebras of functions

Given a topological space (T, τ), where T is the universe of the topology and τ denotes the open sets of the topology, $C(T)$ will denote the set of $[0,1]$-valued continuous functions defined over T, while $C(T, \mathbb{R})$ denotes all real-valued continuous functions defined over T. A *zeroset* $Z \subseteq T$ is a set for which there exists $f \in C(T)$ or $f \in C(T, \mathbb{R})$ (see Lemma 2.2 later on) such that $Z = \{x \in$

$T \mid f(x) = 0\}$. A *cozero* set is a complement of a zeroset, that is a set definable as $\{x \in T \mid f(x) \neq 0\}$ for some continuous function f.

A *Baire set* is a subset of T belonging to the σ-algebra generated by the zerosets of continuous functions from T to $\mathbb{R}$, while a *Borel set* is a subset of T belonging to the σ-algebra generated by the closed sets. We shall denote the σ-algebras of Baire and Borel subsets of T respectively by $\mathcal{B}a(T)$ and $\mathcal{B}o(T)$. As per usual, a Baire function is a function $f\colon T \to \mathbb{R}$ measurable with respect to the spaces $(T, \mathcal{B}a(T))$, $(\mathbb{R}, \mathcal{B}a(\mathbb{R}))$. Borel functions are analogously defined. Note that every Baire set is a Borel set, that is $\mathcal{B}a(T) \subseteq \mathcal{B}o(T)$. It is known that the sets of $[0,1]$-valued Baire and Borel functions defined over $[0,1]^X$ are σ-complete Riesz MV-algebras [11, 7], and they shall be denoted respectively by $\mathrm{Baire}([0,1]^X)$ and $\mathrm{Borel}([0,1]^X)$. We also recall that in both algebras countable suprema are taken pointwise. Indeed, more generally, one can prove that real-valued Baire and Borel functions form Dedekind σ-complete Riesz spaces in which countable suprema are taken pointwise. This is a consequence of the fact that such algebras are spaces of measurable functions, and pointwise countable suprema of measurable functions are measurable (see Problems in [10, Chapter 4]).

Remark 2.1. In the literature there are many inequivalent definitions for the σ-algebra of Baire sets of a space (T, τ). Some of them collapse when T is a locally compact σ-compact Hausdorff space, which covers the cases of interest for us, since our algebras of continuous functions are always defined over compact and Hausdorff spaces. Nonetheless, we stress the fact that the definition we have chosen can be found in [14, Appendix 4A3K], where is also stated that for a metrizable space $\mathcal{B}o(T) = \mathcal{B}a(T)$. Moreover, in [14, Appendix 4A3L] is proved that the given definition of a Baire set is equivalent to the one that can be found in [10, Section 7.1]: $\mathcal{B}a(T)$ is the smallest σ-algebra that makes all bounded continuous functions $f\colon T \to \mathbb{R}$ measurable.

Lemma 2.2. *For every set X, every zeroset of a continuous function from $[0,1]^X$ to $\mathbb{R}$ is the zeroset of some continuous function from $[0,1]^X$ to $[0,1]$.*

Proof. It follows from the remark that the composition of a function valued in $\mathbb{R}$ with the function $\left|\frac{2}{\pi}\arctan(x)\right|$ gives a function valued in $[0,1)$ with the same zeroset. $\square$

We recall that an F_σ is a set that can be written as a countable union of closed sets. A space X is called *basically disconnected* provided that the closure of any open F_σ set is clopen. Note that in literature these spaces are also called *quasi-Stonean* or *Rickart*. As per usual, $\overline{S}$ will denote the topological closure of the set S.

Remark 2.3 (Cozero, open F_σ and countable union of clopens coincide). In Section 5 we will only deal with topological spaces that are compact, Hausdorff, and basically disconnected. Any such space is normal, Tychonoff, and zero-dimensional, see e.g. [28, Theorem 17]. Whence, [12, Corollary 1.5.13] implies that open F_σ sets coincide with cozero sets. Moreover, following [16, Theorem 16.17] (and [23] for a proof in a more modern language), one can see that in a Tychonoff and zero-dimensional space, any cozero set is a countable union of clopens.

Finally, let us recall that for any $A \in \mathbf{RMV}_\sigma$ the set of maximal ideals $Max(A)$ can be endowed with the hull-kernel topology, that is, the topology in which the following sets are a base of open sets:

$$U(a) = \{M \in Max(A) \mid a \in M\}, \quad a \in A.$$

Endowed with this topology, $Max(A)$ becomes a compact, Hausdorff and basically disconnected space. As we mentioned already there is a one-to-one correspondence between algebras in $\mathbf{RMV}_\sigma$ and basically disconnected compact, Hausdorff spaces. This result can be found in [7] and, since it will be the core of the duality of Section 5, we record it in the following proposition.

Proposition 2.4. *The Riesz MV-algebra $C(X)$ is σ-complete if, and only if X is a basically disconnected, compact, Hausdorff space. Thus, for any $R \in \mathbf{RMV}_\sigma$, $Max(R)$ (endowed of the hull-kernel topology) is a basically disconnected, compact, Hausdorff space.*

Moreover, we have the following proposition, that can be recovered from the theory of MV-algebras. We also recall that since $\mathbf{RMV}_\sigma$ is an infinitary variety built on Riesz MV-algebras, it is easily deduced by direct computation that congruences in $\mathbf{RMV}_\sigma$ are in one-one correspondence with σ-ideals, that is, ideals of Riesz MV-algebras that are closed under countable suprema.

Proposition 2.5. *There is a homeomorphism between the maximal space of a σ-complete Riesz MV-algebra A and the maximal space of its Boolean part $B(A)$. The homeomorphism sends M to $M \cap B(A)$.*

Proof. Let us first remark that, for any Riesz MV-algebra A, its maximal ideals are the same as the maximal ideals of its MV-algebraic reduct. Whence, the result follows from [22, Lemma 11.4(ii)], where the claim is proved for σ-complete MV-algebras. $\square$

Finally, we record for future use a kind of compactness for σ-complete ideals. In what follows we shall denote by $\langle B \rangle_\sigma$ the σ-ideal generated by the set B.

Proposition 2.6. *Let A be a σ-complete Riesz MV-algebra. Let $a \in A$, $B \subseteq A$ and $a \in \langle B \rangle_\sigma$. Then there is a countable subset W of B such that $a \in \langle W \rangle_\sigma$.*

Proof. Firstly, we note that we can characterize the σ-ideal generated by B as follows.

$$I = \{x \in A \mid x \leq p(b_1, b_2, \dots, b_n \dots),\ b_n \in B,\ p \text{ is a term without } \neg\}.$$

Indeed, it is easily seen that I is a σ-ideal, since is trivially downward closed, closed to $\oplus$ and to $\bigvee$. Whence $\langle B \rangle_\sigma \subseteq I$. Viceversa, any $x \in I$ is below some $p(b_1, b_2, \dots, b_n \dots)$. Since such p does not contains $\neg$ (and therefore, it does not contain $\bigwedge$), $p(b_1, b_2, \dots, b_n \dots) \in \langle B \rangle_\sigma$, and $I \subseteq \langle B \rangle_\sigma$. Then, for any $a \in \langle B \rangle_\sigma$, if $a \leq p(b_1, b_2, \dots, b_n \dots)$ it is enough to take as W the set of elements $\{b_1, b_2, \dots, b_n \dots\}$ of B that occur in p. $\square$

For better clarity, in the following, boldface letters will denote elements of $[0,1]^X$ as well as tuples of variables to be evaluated there.

3 Free algebras in $\mathbf{RMV}_\sigma$

In [7] the free finitely generated σ-complete Riesz MV-algebras are characterized as algebras of $[0,1]$-valued Borel-measurable functions defined over finite-dimensional hypercubes of type $[0,1]^n$. We start this work with a generalization of this result to free algebras over X, where X is any set. The main result of this section will be a characterization of these free algebras as algebras of Baire functions.

Following the notation set out in Section 2.1, we recall that $IRL(X)$ denotes the algebra of term functions over X variables, and that we shall use the term *IRL-polynomials* for its elements. We also recall that in $IRL(X)$ countable suprema are defined pointwise, since it is an algebra of term functions.

Lemma 3.1. *For every IRL-polynomial $p : [0,1]^X \to [0,1]$, and every Borel subset $B \in \mathcal{B}o([0,1])$, $p^{-1}(B)$ is a Baire set. Whence, every IRL–polynomial is a Baire function.*

Proof. Since IRL-polynomials are term functions, we shall prove the claim by induction on p. Note also that $\mathcal{B}o([0,1])$ can be generated by intervals I of type $[a,1]$ or $[0,a)$, whence we will prove that $p^{-1}(I) \in \mathcal{B}a([0,1]^X)$ for such an I.

For the projections π_i with $i \in X$, the set $\{\mathbf{b} \in [0,1]^X \mid \pi_i(\mathbf{b}) \geq a\}$ is the zeroset of the continuous function $\min(0, \pi_i(\mathbf{x}) - a)$, so it is a Baire set. Since the Baire sets form a σ-algebra, $\pi_i^{-1}(B) \in \mathcal{B}a([0,1]^X)$ for every Borel set B.

For the negation and scalar operations, the inductive step is straightforward.

For the sum $p \oplus q$ it is enough to observe that, for every $a \in [0,1]$, we have $(p \oplus q)(\mathbf{x}) < a$ if, and only if, there is a rational $r < a$ such that $p(\mathbf{x}) < r$ and $q(\mathbf{x}) < a - r$.

For the countable supremum $\bigvee_n p_n$ we have $\bigvee_n p_n(\mathbf{x}) < a$ if, and only if, there is a rational $r < a$ such that $p_n(\mathbf{x}) \leq r$ for every n.

Finally, since $\mathcal{B}a([0,1]) \subseteq \mathcal{B}o([0,1])$ it follows that for any $B \in \mathcal{B}a([0,1])$, $p^{-1}(B)$ is a Baire set of $[0,1]^X$. Consequently, each IRL-polynomial is Baire-measurable, settling the second part of the claim. $\square$

Corollary 3.2. *Every zeroset of an IRL-polynomial in $[0,1]^X$ is a Baire set.*

In the next pages we shall prove that the converse inclusion of Corollary 3.2 holds. An important remark to that end is given by the following proposition.

Proposition 3.3. *For every set X, the zerosets of IRL-polynomials from $[0,1]^X$ to $[0,1]$ form a σ-algebra.*

Proof. The closure under countable intersections follows directly by the presence of the countable supremum operator. To see that zerosets are closed under complements, consider $p(\mathbf{x}) \neq 0$. This holds if, and only if, $\bigvee_n (np(\mathbf{x})) = 1$ if, and only if, $\neg (\bigvee_n (np(\mathbf{x}))) = 0$, whence $[0,1]^X \setminus \{\mathbf{x} \mid p(\mathbf{x}) = 0\} = \{\mathbf{x} \mid \neg (\bigvee_n (np(\mathbf{x}))) = 0\}$, which is the zeroset of $\neg (\bigvee_n np)$. □

Lemma 3.1 also proves that IRL-polynomials are Borel-measurable. In [7] it is shown that the converse of this fact holds when X is finite, that is, for a finite X any Borel-measurable function is an IRL-polynomial. However, this converse fails when X is uncountable. Indeed, we have the following proposition.

Proposition 3.4. *If X is uncountable, there is a Borel function from $[0,1]^X$ to $[0,1]$ which is not an IRL-polynomial.*

Proof. Let us consider a singleton $\{\mathbf{a}\} \subseteq [0,1]^X$. Being $\{\mathbf{a}\}$ closed, its characteristic function χ is a Borel function. However, $\{\mathbf{a}\}$ cannot be the zeroset of an IRL–polynomial. In fact, suppose p is such a polynomial and let V be the set of variables occurring in p. Then V is countable, so every other element of $[0,1]^X$ that has the same coordinates in V as $\mathbf{a}$, is still a zero of p. So, $\{\mathbf{a}\}$ cannot be the zeroset of p. □

Despite not all Borel functions are IRL-polynomials, all continuous functions are so.

Lemma 3.5. *For every set X, every continuous function from $[0,1]^X$ to $[0,1]$ is an IRL-polynomial.*

Proof. The case of a finite X can be found in [7, Corollary 5.7], while for X countably infinite, it follows from the fact that every continuous function of countably many variables $f(x_1, x_2, \ldots, x_n, \ldots)$ is the limit, in $[0,1]^X$, of the functions in finitely many variables $f(x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$, which belongs to $IRL(X)$ since it is σ-complete. Indeed, in general, for any pointwise limit of functions in $IRL(X)$, we have $\lim_n f_n(\mathbf{x}) = \sup_n \inf_{m \geq n} f_m(\mathbf{x}) = \left(\bigvee_n \bigwedge_{m \geq n} f_m\right)(\mathbf{x}) \in IRL(X)$, since $IRL(X)$ is a σ-complete Riesz MV-algebra in which countable suprema and infima are defined pointwise.

For an uncountable X, it is enough to observe that every continuous function f from $[0,1]^X$ to $[0,1]$ depends only on countably many variables, see [10, page 223]. □

Finally, Lemma 3.5 gives the converse of Lemma 3.1 and Corollary 3.2, obtaining the following theorems.

Theorem 3.6. *For every set X, IRL-polynomials and Baire functions defined in $[0,1]^X$ coincide. Whence, the algebra of Baire functions $Baire([0,1]^X)$ is isomorphic to $IRL(X)$, the free σ-complete Riesz MV-algebra over X.*

Proof. The fact that any IRL-polynomial is Baire-measurable was proved in Lemma 3.1. To prove the converse, we recall that Baire functions can be equivalently defined (in our case of compact and Hausdorff topological spaces) by countable iterations of pointwise limits, see [25, Theorem 2.2]. More precisely, Baire functions of class 0 are the continuous functions, Baire functions of class κ are pointwise limits of sequences of Baire functions of class $\alpha < \kappa$. Whence, by Lemma 3.5 every continuous function belongs to $IRL(X)$, which gives the induction basis. If f is a Baire function of class κ, let $\{f_n\}$ be a sequence such that $f(\mathbf{x}) = \lim_n f_n(\mathbf{x})$ for any $\mathbf{x} \in [0,1]^X$. By induction hypothesis each f_n belongs to $IRL(X)$. Thus, $\lim_n f_n(\mathbf{x}) \in IRL(X)$, as in the proof of Lemma 3.5. □

Theorem 3.7. *For every set X, Baire sets and zerosets of IRL-polynomials in $[0,1]^X$ coincide.*

Proof. One inclusion is given in Corollary 3.2. The converse inclusion is a straightforward consequence of Lemma 3.5. Indeed, since every continuous function is an IRL-polynomial, it follows that every zeroset of a continuous function from $[0,1]^X$ to $[0,1]$ is the zeroset of an IRL-polynomial. Whence the σ-algebra $\mathscr{Ba}([0,1]^X)$ is contained in the σ-algebra of zerosets of IRL-polynomials. □

4 On the Marra-Spada duality and the Nullstellensatz

In Theorem 3.7 we proved that Baire sets can be described as zerosets of formulas in an infinitary (but well behaved) propositional logic. This point of view gives a new direction for applications of non-classical logic to real-world problems, and with this in mind we now carry out an investigation of the zerosets of IRL-polynomials from the point of view of algebraic geometry.

Definition 4.1. In what follows, an *IRL-algebraic variety* is the intersection of the zerosets of an arbitrary set of IRL-polynomials, that is, an intersection of Baire sets.

As a straightforward consequence of Proposition 3.3, IRL-algebraic varieties are the closed sets of a topology. This topology will be called $\mathscr{ZIRL}$, which stands for *Zariski IRL-topology*.

Our first result provides a characterization of IRL-algebraic varieties and, therefore, of the topology $\mathscr{ZIRL}$ from a point of view that is inspired by the structure of IRL-polynomials.

Theorem 4.2. *Let X be a set. A subset of $[0,1]^X$ is an IRL–algebraic variety if, and only if, it is defined by a system of inequalities of type $\bigvee_{i\in Y}(x_i \neq a_i)$, where $a_i \in [0,1]$ and Y is a countable subset of X.*

Proof. To settle one direction, we note that every IRL–polynomial p depends only on a countable set $x_1, \ldots, x_n, \ldots$ of variables. So, the zeroset of p is defined by an equation of the form

$$p(x_1, \ldots, x_n, \ldots) = 0,$$

hence it is defined by an intersection of equations

$$(x_1, \dots, x_n, \dots) \in A$$

where $A \subseteq [0,1]^\omega$. If $(a_1, \dots, a_n, \dots)$ ranges over $[0,1]^\omega \setminus A$, then we have an intersection of equations

$$(x_1, \dots, x_n, \dots) \neq (a_1, \dots, a_n, \dots)$$

each of which is a countable disjunction

$$\bigvee_i x_i \neq a_i.$$

Conversely, each countable disjunction of $x_i \neq a_i$ is the zeroset of an IRL-polynomial by Proposition 3.3, so every zeroset of a system of such countable disjunctions is an IRL-algebraic variety. $\square$

Note that, as a consequence of Theorem 4.2, we can characterize the open subsets of $([0,1]^X, \mathcal{ZIRL})$ as the closure under arbitrary unions of sets of type

$$\{\mathbf{x} \in [0,1]^X \mid x_i = a_i \text{ for all } i \in Y\},$$

where Y is a countable subset of X and $a_i \in [0,1]$ for all $i \in Y$.

In [3] the authors prove a very general dual adjunction between the category of (presented) algebras (in an arbitrary variety, even infinitary) and subsets of powers of a fixed algebra A in the variety at hand, with term functions as main characters. In the final part of this section we describe the adjunction in our setting, with $A = [0,1]$, and we obtain dualities for presented and finitely presented σ-complete Riesz MV-algebras. We briefly set the scene.

For any subset $S \subseteq [0,1]^X$, we denote by $\mathbb{I}(S)$ the σ-ideal of IRL-polynomials vanishing on S, that is

$$\mathbb{I}(S) = \{p \in IRL(X) \mid p(\mathbf{x}) = 0 \text{ for any } \mathbf{x} \in S\}.$$

Note that the operator $\mathbb{I}$ is denoted by $\mathbb{C}$ in [3]. Moreover, we shall use $\mathbb{I}(\mathbf{a})$ as a shorthand for $\mathbb{I}(\{\mathbf{a}\})$.

Given a set J of IRL-polynomials in $IRL(X)$, we denote by $\mathbb{V}(J)$ the zeroset of J, that is

$$\mathbb{V}(J) = \{\mathbf{x} \in [0,1]^X \mid p(\mathbf{x}) = 0 \text{ for any } p \in J\} = \bigcap_{p \in J} \mathbb{V}(\{p\}).$$

Note that each $\mathbb{V}(J)$ is an IRL-algebraic variety in the sense of Definition 4.1. Moreover, we shall use $\mathbb{V}(p)$ as a shorthand for $\mathbb{V}(\{p\})$.

These operators give a Galois connection between subsets of points and subsets of polynomials. Moreover, following [3, Section 4], we get the following categories and functors.

1. The category $\mathbf{RMV}^{\mathbf{p}}_{\sigma}$ whose objects are presented σ-complete Riesz MV-algebras and whose arrows are σ-homomorphisms of Riesz MV-algebras. More precisely, an object is a pair $(IRL(X), I)$, where I is a σ-ideal in the free algebra $IRL(X)$. Intuitively, this pair represents the quotient algebra $IRL(X)/I$. Consequently, each morphism $h : (IRL(X), I) \to (IRL(Y), J)$ between pairs is induced by a unique homomorphism $h^p : IRL(X) \to IRL(Y)$ such that $h^p(I) \subseteq J$.

2. The category **Hyper**, whose objects are subsets of hypercubes of type $[0,1]^X$, for an arbitrary X, and arrows are tuples of IRL-polynomials, that is, an arrow in **Hyper** is a map $\eta = (\eta_{y_{|S}})_{y \in Y} \colon S \subseteq [0,1]^X \to T \subseteq [0,1]^Y$, where each η_y belongs to $IRL(X)$. We remark that this definition implies that each $\eta : S \to T$ is the restriction of a tuple of IRL-polynomials $\tilde{\eta} : [0,1]^X \to [0,1]^Y$.

3. The functor $\mathcal{V} : \mathbf{RMV}^{\mathbf{p}}_{\sigma} \to \mathbf{Hyper}$, defined by
   - $\mathcal{V}(IRL(X), J) = \mathbb{V}(J)$;
   - for $h : (IRL(X), J) \to (IRL(Y), K)$, $\mathcal{V}(h) \colon \mathbb{V}(K) \to \mathbb{V}(J)$ is defined as follows. For any $x \in X$, take $p_x \in h^p(\pi_x)$, and note that $p_x \in IRL(Y)$. Now, for $(v_y)_{y \in Y} \in \mathbb{V}(K)$, $\mathcal{V}(h)((v_y)_{y \in Y}) = (p_x((v_y)_{y \in Y}))_{x \in X} \in \mathbb{V}(J)$.

4. The functor $\mathcal{I} : \mathbf{Hyper} \to \mathbf{RMV}^{\mathbf{p}}_{\sigma}$, defined by
   - for $S \subseteq [0,1]^X$, $\mathcal{I}(S) = (IRL(X), \mathbb{I}(S))$,
   - for $\eta = (\eta_y)_{y \in Y} \colon S \subseteq [0,1]^X \to T \subseteq [0,1]^Y$, $\mathcal{I}(\eta) \colon \mathcal{I}(T) \to \mathcal{I}(S)$ is the map given by $f \in IRL(Y) \mapsto f \circ \eta \in IRL(X)$.

Proposition 4.3. *[3, Corollary 4.8] The above defined functors $\mathcal{I}$ and $\mathcal{V}$ are an adjoint pair between the categories $(\mathbf{RMV}^{\mathbf{p}}_{\sigma})^{op}$ and* **Hyper**.

Thus, our goal is to describe the fixed points on both sides of the adjunction and obtain dualities. This will be the content of Corollary 4.16. We shall use the notations set out at the end of Section 2.1.

The following Lemma is [3, Lemma 4.11] specialized to the case of $\mathbf{RMV}_{\sigma}$, where the free algebra is $IRL(X)$, the algebra A in the lemma is $[0,1]$, and σ-ideals correspond to congruences in $\mathbf{RMV}_{\sigma}$.

Lemma 4.4. *[3, Lemma 4.11] For any σ-ideal $I \subseteq IRL(X)$ and any homomorphism $e : IRL(X)/I \to [0,1]$, if e is injective then there exists $\mathbf{a} \in [0,1]^X$ such that $I = \mathbb{I}(\mathbf{a})$.*

Lemma 4.5. *In every free σ-complete Riesz MV-algebra $IRL(X)$, the ideals of type $\mathbb{I}(\mathbf{a})$, with $\mathbf{a} \in [0,1]^X$ are exactly the MV-maximal σ-ideals of $IRL(X)$, that is, the elements of $Max(IRL(X)) \cap Id_{\sigma}(IRL(X))$.*

Proof. For any $M \in Max(IRL(X)) \cap Id_\sigma(IRL(X))$, it is known that the quotient $A := IRL(X)/M$ embeds in $[0,1]$ as a Riesz MV-algebra. Moreover, the quotient A is σ-complete because so is M. Let us denote by $h : A \to [0,1]$ this embedding. Since A is a Riesz MV-algebra, the embedding h is also surjective, indeed for any $\alpha \in [0,1]$, $\alpha = \alpha 1 = \alpha h(1) = h(\alpha 1)$. Thus, h is an order-preserving isomorphism and it is easily proved that it preserves all existing suprema and infima. Consequently, it is a morphism in $\mathbf{RMV}_\sigma$. We are therefore in the hypothesis of Lemma 4.4, and $M = \mathbb{I}(\mathbf{a})$ for some $\mathbf{a}$.

Conversely, each $\mathbb{I}(\mathbf{a})$ is trivially in $Id_\sigma(IRL(X))$ because suprema are defined pointwise in $IRL(X)$. Let us prove that it is maximal. First we notice that it is proper, since the constant function $\mathbf{1}$ does not belong to it. Let I be another ideal such that $\mathbb{I}(\mathbf{a}) \subseteq I$. Suppose there exists $f \in I \setminus \mathbb{I}(\mathbf{a})$. Then $f(\mathbf{a}) > 0$ and there exists $n \in \mathbb{N}$ such that $(nf)(\mathbf{a}) = 1$, where $nf = f \oplus \ldots \oplus f$, n times. Whence $\neg(nf)(\mathbf{a}) = 0$ and $\neg(nf) \in \mathbb{I}(\mathbf{a})$. Thus both f and $\neg(nf)$ belong to I, which implies that $I = IRL(X)$ and $\mathbb{I}(\mathbf{a})$ is maximal. $\square$

Finally, the next theorem characterizes the σ-ideals of $IRL(X)$ that are fixed by the Galois connection $\mathbb{I} - \mathbb{V}$.

Theorem 4.6. *Let J be a subset of IRL-polynomials over a set X of variables. Then:*

(i) $\mathbb{I}(\mathbb{V}(J))$ *is the intersection of all ideals in* $Max(IRL(X)) \cap Id_\sigma(IRL(X))$ *containing* J*;*

(ii) $J = \mathbb{I}(\mathbb{V}(J))$ *if, and only if,* J *is the intersection of the elements of* $Max(IRL(X)) \cap Id_\sigma(IRL(X))$ *that contain it.*

Proof. (i) It is enough to notice that $\mathbb{I}(\mathbb{V}(J)) = \bigcap_{\mathbf{a} \in \mathbb{V}(J)} \mathbb{I}(\mathbf{a})$. By Lemma 4.5 each $\mathbb{I}(\mathbf{a})$ is an MV-maximal σ-ideal. Moreover, $\mathbf{a} \in \mathbb{V}(J)$ if, and only if, $f(\mathbf{a}) = 0$ for any $f \in J$, if, and only if $J \subseteq \mathbb{I}(\mathbf{a})$. Thus the collection $\{\mathbb{I}(\mathbf{a}) \mid \mathbf{a} \in \mathbb{V}(J)\}$ coincides with the set $\{\mathbb{I}(\mathbf{a}) \mid J \subseteq \mathbb{I}(\mathbf{a})\}$ of *all* MV-maximal σ-ideals that contain J.

(ii) It is a straightforward consequence of item (i). $\square$

Each algebra in A in $\mathbf{RMV}_\sigma$ is semisimple as an MV-algebra, which implies that the intersection of $Max(A)$ is trivial. Let us adapt this notion to our framework.

Definition 4.7. An algebra $A \in \mathbf{RMV}_\sigma$ is called σ-semisimple if

$$\bigcap \{M \mid M \in Max(A) \cap Id_\sigma(A)\} = \{0\}.$$

Proposition 4.8. *An algebra $IRL(X)/J \in \mathbf{RMV}_\sigma$ is σ-semisimple if, and only if, J is the intersection of the MV-maximal σ-ideals of $IRL(X)$ that contain J.*

Proof. Let π be the projection modulo J and $A := IRL(X)/J$. We start by noticing that, for any $M \in Max(A) \cap Id_\sigma(A)$, $\pi^{-1}(M)$ is a maximal ideal of $IRL(X)$ by standard results on MV-algebras. More generally, the assignment $N \mapsto N/J$ is an isomorphism of the lattice of ideals of $IRL(X)$ containing J and $Id(IRL(X)/J)$, as MV-algebras. Furthermore, the assignment $N \mapsto N/J$ is also an isomorphism of the lattice of σ-ideals of $IRL(X)$ containing J and $Id_\sigma(IRL(X)/J)$, looked at as infinitary algebras, see [3, Remark 4.16]. Consequently, for any $N \in Max(IRL(X)) \cap Id_\sigma(IRL(X))$, we have $N/J \in Max(A) \cap Id_\sigma(A)$ and viceversa.

If A is σ-semisimple, then $\{0\} = \bigcap\{M \mid M \in Max(A) \cap Id_\sigma(A)\}$. Applying the inverse image of π, we obtain $J = \pi^{-1}(\{0\}) = \bigcap\{\pi^{-1}(M) \mid M \in Max(A) \cap Id_\sigma(A)\}$, which settles the claim.

To prove the converse, we notice that if $\{M_i\}_{i\in I}$ is the family of all MV-maximal σ-ideals of $IRL(X)$ that contain J, then the intersection of $\{\pi(M_i)\}_{i\in I}$ is $\pi(J)$, which is $\{0\}$ in $IRL(X)/J$ and $\{\pi(M_i)\}_{i\in I}$ is the family of all MV-maximal σ-ideals of $IRL(X)/J$. □

Remark 4.9. Let us denote by $\mathcal{M}_\sigma$ the set $Max(IRL(X)) \cap Id_\sigma(IRL(X))$. It is worth noticing that $\mathcal{M}_\sigma \neq Max(IRL(X))$ nor they are homeomorphic topological spaces. Indeed, let J be a proper σ-ideal of $IRL(\kappa)$ such that $\mathbb{V}(J) = \emptyset$, for instance the σ-ideal generated by the characteristic functions of the singletons of $[0,1]^\kappa$ with $\kappa \leq \omega$. Notice that using Proposition 2.6 it is easily seen that $\mathbf{1}$ does not belong to such a J, which is therefore a proper σ-ideal. Then, $\mathbb{I}(\mathbb{V}(J)) = \mathbb{I}(\emptyset) = IRL(\kappa) \neq J$ and consequently $J \neq \bigcap\{M \in \mathcal{M}_\sigma \mid J \subseteq M\}$. Since the quotient ${}^{IRL(\kappa)}/_{J}$ belongs to $\mathbf{RMV}_\sigma$, it is a semisimple Riesz MV-algebra. Thus, as an MV-ideal, J is the intersection of all maximal MV-ideals that contain it and therefore there are maximal ideals that are not σ-complete.

Finally, to see that they are not homeomorphic topological spaces, we recall that $Max(IRL(X))$ is a compact, Hausdorff space and basically disconnected space, while $\mathcal{M}_\sigma$ will inherit the topology $\mathcal{ZIRL}$. If $X = \omega$, this topology is generated by the Borel subsets of $[0,1]^X$ as a base of closed sets and it is not compact in general. Indeed, take any countable set $C = \{\mathbf{x}_n\}_{n\in\mathbb{N}} \subseteq [0,1]^\omega$ and consider $C_m = \{\mathbf{x}_n \mid n \geq m\}$. All of these sets are closed in $\mathcal{ZIRL}$, since they are countable unions of singletons of the Hilbert cube and therefore Borel subsets. Then, any finite intersection of the C_m's is non-empty, while $\bigcap_m C_m = \emptyset$.

The previous remark gives an example of an algebra in $\mathbf{RMV}_\sigma$ that is not σ-semisimple: indeed, it is enough to consider the quotient ${}^{IRL(\kappa)}/_{J}$, where $J = \langle \chi_{\mathbf{a}} \mid \mathbf{a} \in [0,1]^\kappa \rangle_\sigma$.

Proposition 4.10. *Let A be a σ-semisimple σ-complete Riesz MV-algebra and assume that $IRL(X)/I$ is a presentation for A. Then A is isomorphic to $IRL(X)\mid_{\mathbb{V}(I)} = \{g \in [0,1]^{\mathbb{V}(I)} \mid g = f_{|_{\mathbb{V}(I)}}$ for some $f \in IRL(X)\}$.*

Proof. We first notice that in $IRL(X)\mid_{\mathbb{V}(I)}$ countable suprema are defined as in $IRL(X)$, that is, pointwise.

Let us define the function $\eta : IRL(X)/I \to IRL(X) \mid_{\mathbb{V}(I)}$ by $\eta(f/I) = f \mid_{\mathbb{V}(I)}$. It is easily seen that η is a homomorphism of σ-complete Riesz MV-algebra. It is surjective by the very definition of $IRL(X) \mid_{\mathbb{V}(I)}$. Moreover, it is well defined and injective. Indeed, take $f, g \in IRL(X)$, then $f/I = g/I$ if, and only if Chang's distance belongs to I, that is, $d(f,g) \in I$. By hypothesis, $I = \mathbb{I}(\mathbb{V}(I))$ and therefore $d(f,g) \in I$ is equivalent to $d(f,g) = 0$ for any $x \in \mathbb{V}(I)$ which is, in turn, equivalent to $f \mid_{\mathbb{V}(I)} = g \mid_{\mathbb{V}(I)}$. □

Furthermore, we obtain the following universal algebraic characterization of σ-semisimple algebras.

Theorem 4.11. *An algebra $A \in \mathbf{RMV}_\sigma$ is σ-semisimple if, and only if, $A \in ISP([0,1])$.*

Proof. Suppose A is σ-semisimple. Then, by Proposition 4.10, $A \subseteq [0,1]^{\mathbb{V}(I)}$ for a suitable σ-ideal I and therefore $A \in ISP([0,1])$.

Conversely, take $B \in SP[0,1]$. More precisely, suppose $B \subseteq [0,1]^I$ for some set I. Countable suprema in B are inherited from $[0,1]^I$. Then in B, countable suprema are defined pointwise, because so happens in $[0,1]^I$. Whence, consider the ideals $M_i = \{f \in B \mid f(i) = 0\}$, with $i \in I$. With the same argument of Lemma 4.5 it is easily deduced that each M_i is an MV-maximal ideal, and it is a σ-ideal since $f_n(i) = 0$ for every n implies $\left(\bigvee_n f_n\right)(i) = \bigvee_n (f_n(i)) = 0$. Since the intersection of all M_i is zero so is the intersection of $\mathcal{M}_\sigma$, and B is σ-semisimple. Finally, take $A \in ISP([0,1])$, that is, A is an isomorphic copy of some $B \in SP([0,1])$, which is σ-semisimple by the previous argument. It is easily seen by direct computation (using the isomorphism between A and B to reason on the MV-maximal σ-ideals of A and B) that A is σ-semisimple as well, settling the claim. □

Recalling that $\langle S \rangle_\sigma$ denotes the σ-ideal generated by a set $S \subseteq IRL(X)$, we have the following result.

Lemma 4.12. *Let $p, q \in IRL(X)$. Then $\mathbb{V}(q) \subseteq \mathbb{V}(p)$ if, and only if, $\langle p \rangle_\sigma \subseteq \langle q \rangle_\sigma$.*

Proof. For the non-trivial direction, let us prove that $\mathbb{V}(q) \subseteq \mathbb{V}(p)$ implies $p \leq \bigvee_n nq$.

Assume by way of contradiction that $p \not\leq \bigvee_n nq$, whence there exists $\mathbf{a} \in [0,1]^X$ such that $p(\mathbf{a}) \not\leq \left(\bigvee_n nq\right)(\mathbf{a})$ and therefore $\left(\bigvee_n nq\right)(\mathbf{a}) < p(\mathbf{a})$. Since $\mathbb{V}(q) \subseteq \mathbb{V}(p)$, it must be $q(\mathbf{a}) > 0$, for otherwise we would have $p(\mathbf{a}) = \left(\bigvee_n nq\right)(\mathbf{a}) = 0$. But $q(\mathbf{a}) > 0$ implies $\left(\bigvee_n nq\right)(\mathbf{a}) = 1 < p(\mathbf{a}) \leq 1$, a contradiction. Whence, $p \leq \bigvee_n nq$ implies $p \in \langle q \rangle_\sigma$ and therefore $\langle p \rangle_\sigma \subseteq \langle q \rangle_\sigma$. □

Remark 4.13. We note that the more general version of Lemma 4.12, that is

$$\text{“ } \mathbb{V}(I) \subseteq \mathbb{V}(J) \text{ if, and only if, } J \subseteq I, \text{ for } I, J \in Id_\sigma(IRL(X)) \text{ ”}$$

does not hold true. Indeed, take $IRL([0,1])$ and $J = \langle \chi_a \mid a \in [0,1] \rangle_\sigma$. We have that $\mathbb{V}(\mathbf{1}) = \mathbb{V}(J) = \emptyset$ but $\mathbf{1} \notin J$.

Lemma 4.14. *For any $p \in IRL(X)$, $\langle p\rangle_\sigma = \mathbb{I}(\mathbb{V}(p))$.*

Proof. The inclusion $\langle p\rangle_\sigma \subseteq \mathbb{I}(\mathbb{V}(p))$ is always true, while the converse inclusion follows from Lemma 4.12. Indeed, for any $q \in \mathbb{I}(\mathbb{V}(p))$ it holds that $q(\mathbf{a}) = 0$ for any $\mathbf{a} \in \mathbb{V}(p)$. That is, $\mathbb{V}(p) \subseteq \mathbb{V}(q)$. Whence, $q \in \langle p\rangle_\sigma$ by Lemma 4.12. □

Theorem 4.6 will help us in describing the fixed points on one side of the adjunction. On the other side, we can prove the following.

Theorem 4.15. *Let S be a subset of $[0,1]^X$. Then $\mathbb{V}(\mathbb{I}(S))$ is the intersection of all Baire subsets of $[0,1]^X$ containing S. Thus, $\mathbb{V}(\mathbb{I}(S)) = S$ if, and only if, S is an IRL-algebraic variety.*

Proof. We have that $\mathbb{V}(\mathbb{I}(S)) = \bigcap_{p\in\mathbb{I}(S)} \mathbb{V}(p) = \bigcap_{\mathbb{V}(p)\subseteq S} \mathbb{V}(p)$. Whence, the result follows from the remark that, from Theorem 3.7, the sets of type $\mathbb{V}(p)$ are exactly the Baire subsets of $[0,1]^X$. We also remark that IRL-algebraic varieties were defined exactly as arbitrary intersections of zerosets in Definition 4.1, that is, as arbitrary intersections of Baire sets by Theorem 3.7. □

Whence, by restricting the adjunction we obtain the following duality. Note that our objects are formally *embedded* IRL-algebraic varieties, meaning intersections of Baire sets $S \hookrightarrow [0,1]^X$.

Corollary 4.16. *The adjunction $(\mathcal{I}, \mathcal{V})$ restricts to a duality between the full subcategory $\mathbf{ssRMV}^{\mathbf{p}}_\sigma$ of $\mathbf{RMV}^{\mathbf{p}}_\sigma$ whose objects are σ-semisimple algebras, and the full subcategory* **IRL** *of* **Hyper** *whose objects are IRL-algebraic varieties.*

Proof. On one side, $\mathcal{V}(\mathcal{I}(S)) = S$ if, and only if $\mathbb{V}(\mathbb{I}(S)) = S$, which is given by Theorem 4.15. For the other, we have that $\mathcal{I}(\mathcal{V}(IRL(X), J)) = (IRL(X), J)$ if, and only if $J = \mathbb{I}(\mathbb{V}(J))$. From Theorem 4.6 and Proposition 4.8 we can deduce that this holds if, and only if, the quotient $IRL(X)/J$ is σ-semisimple. □

Furthermore, with the aim of capturing Baire subsets in hypercubes of at most countable dimension, we restrict to the case of *countably presented* algebras. By countably presented algebra here we mean a σ-complete RMV-algebra that is presented by $(IRL(X), J)$, where $J = \langle p\rangle_\sigma$ is a principal ideal and X is countable. If X is a finite set, the algebra will be called *finitely presented*. Note that in our setting, principal and countably generated σ-ideals coincide, indeed it is straightforward to prove that $\langle \{p_n\}_{n\in\mathbb{N}}\rangle_\sigma = \left\langle \bigvee_{n\in\mathbb{N}} p_n\right\rangle_\sigma$.

Corollary 4.17. *The adjunction $(\mathcal{I}, \mathcal{V})$ restricts to a duality between the full subcategory $\mathbf{RMV}^{\omega}_\sigma$ of countable presented and σ-complete RMV-algebras and full subcategory $\mathbf{Baire}^{\omega}$ of* **Hyper** *whose objects are Baire subsets in hypercubes of at most countable dimension. Finitely presented algebras correspond to Baire subsets of finite-dimensional hypercubes.*

Proof. We have $\mathcal{I}(\mathcal{V}(IRL(X), \langle p\rangle_\sigma)) = (IRL(X), \langle p\rangle_\sigma)$ by Lemma 4.14 and, if S is a Baire set, $S = \mathbb{V}(\mathbb{I}(S))$ by Theorem 4.15. Furthermore, $IRL(X)/\langle p\rangle_\sigma$ corresponds to the zeroset of p, which is a Baire set by Theorem 3.7. Conversely, again by Theorem 3.7, every Baire set $S \subseteq [0,1]^X$ is the zeroset of an IRL polynomial p, so S corresponds to the countably presented σ-complete RMV-algebra $(IRL(X), \langle p \rangle_\sigma)$. $\square$

To close this section, we mention that in the literature one can find the notion of σ-completion of a Riesz space. Since $IRL(X)$ is σ-complete, one can wonder if it coincides with the σ-completion of $RL(X)$, the Lindenbaum-Tarski algebra of the logic of Riesz MV-algebras. Equivalently, it is left open to inquire if $Max(IRL(X))$ is the so-called σ-cover of $Max(RL(X))$, that is the smallest basically disconnected cover of $Max(RL(X))$ in the sense of, e.g., [9].

5 On the Kakutani duality

The Gelfand-Naimark-Yosida-Krein-Kakutani duality is a fundamental duality that carries many names, since it has been investigated in different settings by different authors. For example, when one deals with C^*-algebras, the duality is usually referred to as Gelfand's duality. In the setting of norm-complete vector lattices (or abstract M-spaces), it carries the name of Kakutani [18]. This duality can be described by a functor that maps each compact and Hausdorff space X into the Riesz space $C(X, \mathbb{R})$ and each continuous function $f : X \to Y$ into $\tilde{f} : C(Y) \to C(X)$ defined by $h \mapsto h \circ f$. Conversely, to each abstract M-space R it is associated the compact Hausdorff space $Max(R)$ of its maximal ideals. It is restricted to norm-complete Riesz MV-algebras in [8], where morphisms are taken to be homomorphisms of Riesz MV-algebras.

Since any σ-complete Riesz MV-algebra is norm-complete, in this section we shall describe the duality restricted to the algebraic category $\mathbf{RVM}_\sigma$, of σ-complete Riesz MV-algebras equipped with σ-complete homomorphisms, that is homomorphisms that preserve countable suprema. Note that Proposition 2.4 completely describes the duality on the level of objects, whence our aim is to characterize those continuous functions that correspond, via Kakutani's duality, to σ-homomorphisms of Riesz MV-algebras.

Since we will prove the result passing through Boolean algebras, we also note that in [26, Section 22] the author discusses Boolean algebras closed under infinite joins and meets of arity an arbitrary $\mathfrak{m}$, and restricts Stone duality to the case of $\mathfrak{m}$-complete Boolean algebras. Let us start with a definition. Let $f : X \to Y$ be a continuous function between two compact, Hausdorff and basically disconnected spaces. A subset of Y is called σ-closed if it is a countable intersection of clopens. A subset V of Y is called σ-nowhere dense if it is a subset of a σ-closed and nowhere dense subset of Y. Finally, f is called σ-continuous if $f^{-1}(V)$ is σ-nowhere dense for any σ-nowhere dense set V.

Theorem 5.1. *[26, 22.5] The algebraic category $\mathbf{BA}_\sigma$, of σ-complete Boolean algebras, is dual to the category* **BDKH***, whose objects are basically disconnected, compact, Hausdorff spaces and whose morphisms are σ-continuous functions.*

In what follows, unless otherwise specified, X and Y will always denote compact, Hausdorff and basically disconnected spaces. For any continuous $f : X \to Y$ let $\tilde{f} : C(Y) \to C(X)$, $h \mapsto h \circ f$, be the corresponding homomorphism of Riesz MV-algebras. Note that, $\tilde{f}$ being a homomorphism of Riesz MV-algebras, it sends the Boolean part of $C(Y)$ to the Boolean part of $C(X)$ and it preserves the scalar multiplication.

Lemma 5.2. *If $\{C_n\}_{n\in\mathbb{N}}$ is a countable sequence of clopens in a compact, Hausdorff, basically disconnected space X, then $\bigvee_n \chi_{C_n} = \chi_{\overline{U}}$, where $U = \bigcup_n C_n$ and the supremum is taken in $C(X)$.*

Proof. By definition, U is an F_σ set and it is open, whence $\overline{U}$ is a clopen since the space is compact, Hausdorff and basically disconnected. Hence the characteristic function $\chi_{\overline{U}}$ is a continuous function. Note that it is enough to prove that $\chi_{\overline{U}}$ is the supremum in the Boolean part of $C(X)$, whence let us consider another clopen set K such that $\chi_{C_n} \leq \chi_K$ for every n. Then $C_n \subseteq K$, hence $\bigcup_n C_n \subseteq K$ and $U \subseteq K$, which implies that $\overline{U} \subseteq K$, so $\chi_{\overline{U}} \leq \chi_K$. $\square$

Lemma 5.3. *Let $f : X \to Y$ be a continuous function, and $\tilde{f} : C(Y) \to C(X)$ the dual arrow between Riesz MV-algebras. Then $\tilde{f}$ is a σ-homomorphism on the Boolean part of $C(Y)$ if, and only if, for every countable union U of clopens of Y, we have*

$$f^{-1}(\overline{U}) = \overline{f^{-1}(U)}. \tag{1}$$

Proof. Take a sequence of clopens C_n in Y, and denote by U the union, that is, $U = \bigcup_n C_n$. Then $\tilde{f}$ is a σ-homomorphism on the Boolean part of $C(Y)$ if, and only if,

$$\tilde{f}\left(\bigvee_n \chi_{C_n}\right) = \bigvee_n \tilde{f}\left(\chi_{C_n}\right).$$

By Lemma 5.2 this is equivalent to

$$\tilde{f}\left(\chi_{\overline{U}}\right) = \bigvee_n \tilde{f}(\chi_{C_n}).$$

Using the definition of $\tilde{f}$, the latter can be rewritten as

$$\chi_{\overline{U}} \circ f = \bigvee_n (\chi_{C_n} \circ f),$$

which, via some easy computation and Lemma 5.2, gives

$$\begin{aligned} \chi_{f^{-1}(\overline{U})} &= \bigvee_n (\chi_{C_n} \circ f) = \bigvee_n \chi_{f^{-1}(C_n)} = \chi_{\overline{\bigcup_n f^{-1}(C_n)}} = \\ &= \chi_{\overline{f^{-1}(\bigcup_n C_n)}} = \chi_{\overline{f^{-1}(U)}}. \end{aligned}$$

Thus, $\tilde{f}$ is a σ-homomorphism on the Boolean part of $C(Y)$ if, and only if, $\chi_{f^{-1}(\overline{U})} = \chi_{\overline{f^{-1}(U)}}$, which is equivalent to the equality $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$, settling the claim. $\square$

Since cozero sets coincide with countable unions of clopens, see Remark 2.3, we introduce the following definition.

Definition 5.4. Let X, Y be compact and Hausdorff topological spaces. We will call a function $f : X \to Y$ *cozero-closed* if, for any cozero set U, $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$, that is, if it satisfies Equation (1).

Corollary 5.5. *Let $f : X \to Y$ be a continuous function between compact, Hausdorff basically disconnected spaces. Then f is cozero-closed if, and only if, it is σ-continuous.*

Proof. It follows from Lemma 5.3 together with Theorem 5.1. Indeed, let us denote by f^* the function that corresponds to f via Stone duality. Then, for any $C \in Clop(Y) \simeq C(Y, \{0,1\})$, we have $f^*(C) = f^{-1}(C)$, see [26, Section 11]. Consequently, f is cozero-closed if, and only if, $\tilde{f}$ is a σ-homomorphism on the Boolean part of $C(Y)$ if, and only if, f^* is a σ-homomorphism of Boolean algebras if, and only if, f is σ-continuous. $\square$

Thus, in order to characterize the arrows that are dual to σ-homomorphisms in $\mathbf{RMV}_\sigma$, we will prove that Lemma 5.3 extends to the whole $C(Y)$. Let us start with two preliminary lemmas.

Lemma 5.6. *Every function of $C(Y)$ is a countable supremum of rational multiples of Boolean elements of $C(Y)$. Hence every function of $C(Y)$ is the supremum of all rational multiples of Boolean elements below it.*

Proof. Let $g \in C(Y)$. Let q be a rational between 0 and 1. Then $g^{-1}(]q, 1])$ is open and F_σ, so its closure $\overline{g^{-1}(]q, 1])}$ is clopen since Y is basically disconnected.

Now we have

$$g = \bigvee_q q\chi_{\overline{g^{-1}(]q,1])}}.$$

In fact, if $y \in \overline{g^{-1}(]q, 1])}$, then $g(y) \geq q$, so $g \geq q\chi_{\overline{g^{-1}(]q,1]}}$ for every q.

Conversely, take $y \in Y$. Let us show that the value of the supremum in y is at least $g(y)$. If $g(y) = 0$ this is clear. Otherwise, if $g(y) \neq 0$, let q' be a rational such that $g(y) > q'$. Then the value of $q'\chi_{\overline{g^{-1}(]q',1]}}$ in y is q', so the value of $\bigvee_q q\chi_{\overline{g^{-1}(]q,1]}}$ in y is at least q'. Letting q' tend to $g(y)$, the value of this supremum in y is at least $g(y)$. This proves that the supremum is at least g.

$\square$

Lemma 5.7. *Let λ_n be a sequence of rational numbers and χ_n be a sequence of Boolean elements in $C(Y)$. If $\tilde{f}$ is a σ-homomorphism on the Boolean part of $C(Y)$, then $\tilde{f}\left(\bigvee_n \lambda_n\chi_n\right) = \bigvee_n \tilde{f}\left(\lambda_n\chi_n\right)$.*

Proof. Let λ be any rational and χ be any Boolean element of $C(Y)$. By the previous lemma it is enough to show that the condition

$$\lambda\chi \leq \tilde{f}\left(\bigvee_n \lambda_n\chi_n\right) \tag{2}$$

implies the condition

$$\lambda\chi \leq \bigvee_n \tilde{f}\left(\lambda_n\chi_n\right). \tag{3}$$

Suppose (2) and let us prove (3). Note that we can suppose $\lambda > 0$. Fix a real $\epsilon > 0$. Let A be the set of n such that $\lambda_n < \lambda - \epsilon$ and B the set of n such that $\lambda_n \geq \lambda - \epsilon$.

Then from Equation (2) we have

$$\begin{aligned}\lambda\chi \leq & \tilde{f}\left(\bigvee_{n\in A}\lambda_n\chi_n\right) \vee \tilde{f}\left(\bigvee_{n\in B}\lambda_n\chi_n\right)\\ \leq & (\lambda-\epsilon)\tilde{f}\left(\bigvee_{n\in A}\chi_n\right) \vee \tilde{f}\left(\bigvee_{n\in B}\lambda_n\chi_n\right).\end{aligned}$$

Since χ is a Boolean function, the values of $\lambda\chi$ are all zero or λ, so in the inequality above we can eliminate the term $(\lambda-\epsilon)\tilde{f}\left(\bigvee_{n\in A}\chi_n\right)$, whose values are all less than λ, and we obtain

$$\lambda\chi \leq \tilde{f}\left(\bigvee_{n\in B}\lambda_n\chi_n\right). \tag{4}$$

We also recall that, in our framework, rationals are taken in $[0,1]$.

Moreover we have $\chi \leq \bigvee_{n\in B}\tilde{f}(\chi_n)$. In fact, suppose $\chi(x) = 1$ but $\bigvee_{n\in B}\tilde{f}(\chi_n)(x) = 0$. Then $\tilde{f}(\bigvee_{n\in B}\chi_n)(x) = 0$ and $\tilde{f}(\bigvee_{n\in B}\lambda_n\chi_n)(x) = 0$, whereas $\lambda\chi(x) = \lambda > 0$, contrary to (4).

By multiplying with $\lambda - \epsilon$ we have

$$(\lambda-\epsilon)\chi \leq \bigvee_{n\in B}\tilde{f}((\lambda-\epsilon)\chi_n) \leq \bigvee_{n\in B}\tilde{f}(\lambda_n\chi_n) \leq \bigvee_n \tilde{f}(\lambda_n\chi_n)$$

and letting ϵ tend to zero, we have (3). □

Theorem 5.8. *If $\tilde{f}$ is a σ-homomorphism on the Boolean part of $C(Y)$, then it is a σ-homomorphism on $C(Y)$.*

Proof. Suppose $\tilde{f}$ is a σ-homomorphism on the Boolean part of $C(Y)$. Let g_n be a sequence of elements of $C(Y)$. By Lemmas 5.6 and 5.7 we have $g_n = \bigvee_m \lambda_{n,m}\chi_{n,m}$ and

$$\begin{aligned}\tilde{f}\left(\bigvee_n g_n\right) = & \tilde{f}\left(\bigvee_n\bigvee_m \lambda_{n,m}\chi_{n,m}\right) = \\ & = \bigvee_n\bigvee_m \tilde{f}\left(\lambda_{n,m}\chi_{n,m}\right) = \bigvee_n \tilde{f}\left(\bigvee_m \lambda_{n,m}\chi_{n,m}\right) = \bigvee_n \tilde{f}(g_n),\end{aligned}$$

settling the claim. □

Summing up the previous results we deduce the following theorem.

Theorem 5.9. *Let $f : X \to Y$ be a continuous function, and $\tilde{f} : C(Y) \to C(X)$ the dual arrow between Riesz MV-algebras. Then $\tilde{f}$ is a σ-homomorphism on $C(Y)$ if, and only if, it is a σ-homomorphism on its Boolean part.*

As a consequence of the previous theorem, together with Proposition 2.5, we obtain a duality and an equivalence for the category of σ-complete Riesz MV-algebras and σ-homomorphisms of Riesz MV-algebras.

Theorem 5.10. *The algebraic category $\mathbf{RMV}_\sigma$, whose objects are σ-complete Riesz MV-algebras, is*

(i) dual to the category **BDKH** *whose objects are basically disconnected, compact, Hausdorff spaces and whose morphisms are continuous and cozero-closed functions.*

(ii) equivalent to the algebraic category $\mathbf{BA}_\sigma$, of σ-complete Boolean algebras.

As mentioned at the beginning of this section, the duality we have described is one version of a much deeper result. Whence, we close this section by linking σ-complete Riesz MV-algebras to a subcategory of commutative C^*-algebras. Note that the theory of MV-algebras was already connected to AF C^*-algebras by D. Mundici in [21].

We recall that a C^*-algebra is a Banach algebra $(A, \|\cdot\|)$ over the complex field $\mathbb{C}$, equipped with an involution * that satisfies the equation $\|xx^*\| = \|x\|^2$. For further details, please see [2] and [17, Section 5]. A projection in A is a self-adjoint and idempotent element of A, that is an element e such that $e^* = e$ and $e^2 = e$.

By Gelfand's duality, any unital and commutative C^*-algebra A is isomorphic to the algebra of complex-valued continuous functions $C(X, \mathbb{C})$, for a compact and Hausdorff space X, usually called the *Gelfand spectrum* of A. On arrows, the duality maps $*$-homomorphisms, that is homomorphisms that preserve the whole structure of C^*-algebra, into continuous functions. Moreover, we recall that for unital and commutative C^*-algebras, the set of projections $Proj(A)$ is isomorphic to the Boolean algebra of clopen sets of X. For our purposes, let us define *Rickart C^*-algebras* as those C^*-algebras such that the annihilator of each element is generated by a projection of A. That is, if we denote by $ann(x) = \{a \in A \mid xa = 0\}$ the annihilator of an element x, then $ann(x) = eA$, for some $e \in Proj(A)$.

Theorem 5.11. *[2, Theorem 1.8.1] Let A be a unital and commutative C^*-algebra and let X be its Gelfand spectrum. Then, A is a Rickart C^*-algebra if, only if, X is a compact, Hausdorff and basically disconnected space.*

On arrows, since $Clop(X) \simeq Proj(A)$, we can deduce the following theorem. Note that, as in the real-valued case, the functor that gives Gelfand's duality works on continuous functions by pre-composition.

Theorem 5.12. *A map $f : X \to Y$ between two objects X, Y of* **BDKH** *is cozero-closed if, and only if, $\tilde{f} : C(Y, \mathbb{C}) \to C(X, \mathbb{C})$ preserves suprema of countable sets of projections.*

Proof. Since $Clop(X) \simeq Proj(C(X, \mathbb{C}))$, any projection $e \in Proj(C(X, \mathbb{C}))$ corresponds to a $C \in Clop(X)$ and therefore to a $\chi_C \in C(X, \mathbb{C})$. Whence, the preservation of countable suprema of projections is equivalent to the condition of Lemma 5.3. $\square$

Let us denote by $\mathbf{ucRC}^*$ the category whose objects are unital and commutative Rickart C^*-algebras and whose morphisms are $*$-homomorphisms that preserve countable suprema of projections.

Corollary 5.13. *The categories* **BDKH** *and* $\mathbf{ucRC}^*$ *are dually equivalent. Whence, the categories* $\mathbf{ucRC}^*$ *and* $\mathbf{RMV}_\sigma$ *are equivalent.*

Note that, as a consequence of Corollary 5.13 and of the definition of the functors, any σ-complete Riesz MV-algebra is the unit interval of the real part of a Rickart C^*-algebra. Indeed, by duality, to the same object X of **BDKH** we associate the objects $C(X)$ of $\mathbf{RMV}_\sigma$ and $C(X, \mathbb{C})$ of $\mathbf{ucRC}^*$.

Moreover, there is a unital and commutative Rickart C^*-algebra whose real part is isomorphic to the algebra of Baire functions over $[0, 1]^X$. Denoted by T the space $Max(IRL(X))$, this Rickart C^*-algebra is given by $C(T, \mathbb{C})$.

Corollary 5.14. *The algebra of real-valued Baire functions $Baire([0, 1]^X)$ is isomorphic to the Rickart C^*-algebra $C(T, \mathbb{C})$.*

To close the section, we draw a diagram to depict all dualities and equivalences enlightened here. The full lines represent the dualities, while the dashed lines are equivalences.

$$
\begin{tikzcd}
\mathbf{RMV}_\sigma \arrow[r, leftrightarrow, dashed] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow, dashed] & \mathbf{BA}_\sigma \arrow[d, leftrightarrow, dashed] \arrow[dl, leftrightarrow] \\
\mathbf{BDKH} \arrow[r, leftrightarrow] & \mathbf{ucRC}^*
\end{tikzcd}
$$

6 On polynomial completeness

The notion of *polynomial completeness* was introduced in [1] to classify those MV-algebras in which the two notions of MV-polynomial functions (that is, term functions) and polynomial coincide. This subtle difference in notions is quite crucial when one wishes for a free algebra to behave like a "true" algebra of polynomials, and it was indeed at the core of Section 6 of [1]. Since we are interested in the applicability of term functions in $\mathbf{RMV}_\sigma$ outside the realm of logic, see the end of Section 1, in this section we shall analyze polynomial completeness within Riesz MV-algebras and Boolean algebras. To do so, let us recall formally the definition in the case of an arbitrary variety of MV-algebras, with operations at most of countable arity.

Definition 6.1. Let A be an algebra in a variety $\mathbf{V}$ of algebras that have an MV-reduct. Denoted by $\mathbf{x}$ a countable set of variables $\{x_1, \ldots, x_n, \ldots\}$, and denoted by $A[\mathbf{x}]$ the algebra of polynomials with variables taken from $\mathbf{x}$ and coefficients in A, we will call A *polynomially complete* if the only polynomial $p \in A[\mathbf{x}]$ inducing the zero-function on A is the zero polynomial. Equivalently, A is polynomially complete if every polynomial that induces the zero function on A induces the zero function on any $B \supseteq A$, with $B \in \mathbf{V}$. For a more precise definition of the algebra $A[\mathbf{x}]$ we urge the reader to consult [1, Section 3.2].

We remark that this notion depends on the variety $\mathbf{V}$, but we shall simply talk about polynomially complete algebras when the variety is clear from context. As an example, in $\mathbf{V} = \mathbf{MV}$, $A = \{0, 1\}$ is not a polynomially complete MV-algebra since the polynomial $p(x) = x \wedge \neg x$ induces the zero function in A, but it does not in $[0, 1] \supseteq A$. More generally, one can see that all finite MV-chains are not polynomially complete, while $[0, 1]$ was proved to be polynomially complete in [1, Proposition 6.3].

For a fixed algebra $A \in \mathbf{V}$, we shall think of an extension $B \supseteq A$ as a pair (B, h), with $h : A \to B$ a homomorphism in $\mathbf{V}$.

Remark 6.2. Note that we can think of a $p \in A[\mathbf{x}]$ as a $p(\mathbf{a}, \mathbf{x})$, where $p(\mathbf{y}, \mathbf{x})$ is a term in the language of A with two sets of countable variables and $\mathbf{a}$ is the sequence of the elements from A occurring in p in which we evaluate the variables from $\mathbf{y}$. Thus, in the following, the notation $p(\mathbf{a}, \mathbf{x})$ stands for: $p \in A[\mathbf{x}]$ and $\mathbf{a} = \{a_1, \ldots, a_m, \ldots\} \subseteq A$ are the constants occurring in p. When needed for computations, we shall write variables and constants explicitly.

As remarked above, not all MV-algebras are polynomially complete. In this section we see that all Riesz MV-algebras as well as all Boolean algebras are well behaved with respect to this notion, while we settled on a weak version of polynomial completeness for σ-semisimple σ-complete Riesz MV-algebras.

Definition 6.3 (Local polynomial completeness)**.** Let $\mathbf{C} \subseteq \mathbf{RMV}_\sigma$ a class of σ-complete Riesz MV-algebras. We say that $A \in \mathbf{RMV}_\sigma$ is *locally* polynomially complete with respect to $\mathbf{C}$ if every polynomial that induces the zero function on A induces the zero function on any $B \supseteq A$, with $B \in \mathbf{C}$.

Theorem 6.4. *Every σ-semisimple σ-complete Riesz MV-algebra is locally polynomially complete with respect to $ISP([0, 1])$.*

Proof. We use the characterizations of σ-semisimple σ-complete Riesz MV-algebras given in Theorem 4.11 and Proposition 4.10. Take $A, B \in SP([0, 1])$, with $B \supseteq A$. Let $V \subseteq [0, 1]^X$ and $W \subseteq [0, 1]^Y$ be the IRL-algebraic varieties such that $A = IRL(X)|_V$ and $B = IRL(Y)|_W$. Then every homomorphism from A to B has the form $h_\eta(a) = a \circ \eta$, where $\eta : W \to V$ is the IRL-map given by Corollary 4.16. Furthermore, up to isomorphism, all operations are computed pointwise in any algebra of $SP([0,1])$.

Towards a contradiction let $p(\mathbf{a},\mathbf{x})$ be a polynomial with countable coefficients in A and which is zero in A but nonzero in $B \supseteq A$. Let $h : A \to B$. Then there are $b_1, \ldots, b_n, \ldots \in B$ such that $p(a_1 \circ \eta, \ldots, a_m \circ \eta, \ldots, b_1, \ldots, b_n, \ldots) \neq 0$. There is $y \in W$ such that

$$p\left((a_1 \circ \eta)(y), \ldots, (a_m \circ \eta)(y), \ldots, b_1(y), \ldots, b_n(y), \ldots\right) \neq 0.$$

Let $x = \eta(y)$ and let ϵ_i be the constant function $b_i(y)$ on V, which is in A because every constant function belongs to $IRL(X)$. Then

$$p(a_1(x), \ldots, a_m(x), \ldots \epsilon_1(x), \ldots, \epsilon_n(x), \ldots) \neq 0$$

hence

$$p(a_1, \ldots, a_m, \ldots, \epsilon_1, \ldots, \epsilon_n, \ldots) \neq 0.$$

Whence p is not zero on A, and this is a contradiction. So A is locally polynomially complete. $\square$

On Boolean algebras we have the following result.

Theorem 6.5. *Every Boolean algebra is polynomially complete.*

Proof. Let A be a Boolean algebra and, by way of contradiction, assume that $p(a_1, \ldots, a_m, x_1, \ldots, x_n)$ is a polynomial with coefficients in A which is zero in A but nonzero in some $B \supseteq A$. Let $f : A \to B$. By Stone duality we have $A = C(X, \{0,1\})$, $B = C(Y, \{0,1\})$ and there is a function $\pi : Y \to X$ such that $f(a) = a \circ \pi$ for any $a \in A$.

Towards a contradiction let $p(\mathbf{a},\mathbf{x})$ be a polynomial with coefficients in A and which is zero in A but nonzero in some $B \supseteq A$. Let $f : A \to B$. Then there are $b_1, \ldots, b_n \in B$ such that $p(a_1 \circ \pi, \ldots, a_m \circ \pi, b_1, \ldots, b_n) \neq 0$. Now the proof strategy is analogous to the one of Theorem 6.4. $\square$

Finally, any Riesz MV-algebra is polynomially complete.

Theorem 6.6. *All Riesz MV-algebras are polynomially complete.*

Proof. Let B be an A-algebra, with $f : A \to B$ a homomorphism of Riesz MV-algebras. It follows from [20, Theorem 3.5] that there exists an ultrapower $[0,1]^*$ of $[0,1]$ and a set Y such that $B \subseteq ([0,1]^*)^Y$.

Suppose $p(a_1, \ldots, a_m, x_1, \ldots, x_n)$ is a zero polynomial in A but not in B. Then for some $b_1, \ldots, b_n \in B$ we have

$$p(f(a_1), \ldots, f(a_m), b_1, \ldots, b_n) \neq 0.$$

Since $B \subseteq ([0,1]^*)^Y$, for some $y \in Y$ we have

$$p(f(a_1)(y), \ldots, f(a_m)(y), b_1(y), \ldots, b_n(y)) \neq 0.$$

Note that $b_i(y) \in [0,1]^*$. The idea is to replace each $b_i(y)$ with a constant function whose value is a real number, in the order $i = 1, \dots, n$.

Suppose $b_1(y)$ is infinitesimally close to a real number β.

If $p(f(a_1)(y), \dots, f(a_m)(y), \beta, \dots, \dots, b_n(y)) \neq 0$ we can replace $b_1(y)$ with the constant β. If instead $p(f(a_1)(y), \dots, f(a_m)(y), \beta, \dots, \dots, b_n(y)) = 0$ then the left or right derivative of the function

$$P(x) = p(f(a_1)(y), \dots, f(a_m)(y), x, \dots, b_n(y))$$

is nonzero in $x = \beta$. These derivatives are real numbers. So, there is a real interval I having β as minimum or maximum such that $P(x) \neq 0$ at the interior of I. We can replace $b_1(y)$ with the constant function whose value is any real number lying at the interior of I. Then we replace $b_2(y), b_3(y)$ and so on until we arrive at

$$p(f(a_1)(y), \dots, f(a_m)(y), f(a'_1)(y), \dots, f(a'_n)(y)) \neq 0$$

where $a'_1, \dots, a'_n$ are constant functions in A. Note that f is a homomorphism of Riesz MV-algebras, so it maps constants to constants.

Hence

$$p(f(a_1), \dots, f(a_m), f(a'_1), \dots, f(a'_n)) \neq 0$$

and since f is a homomorphism

$$p(a_1, \dots, a_m, a'_1, \dots, a'_n) \neq 0$$

and p is not zero on A, a contradiction. So A is polynomially complete. □

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