A general view on normal form theorems for Łukasiewicz logic with product

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Abstract

In this survey paper we explore the connection between the Pierce-Birkhoff conjecture and Łukasiewicz logic with product. Conservative extensions of Łukasiewicz logic can be defined by adding an internal product or a multiplication with scalars from $[0, 1]$. The corresponding models reflect an algebraic hierarchy of lattice-ordered structures, from groups to algebras. We prove a general version of the normal form theorem and we state a local version of the Pierce-Birkhoff conjecture.
Keywords: Łukasiewicz logic, MV-algebras, normal form theorem, Pierce-Birkhoff conjecture.

1 Introduction

The modern evolution of Łukasiewicz logic, defined in [Łukasiewicz and Tarski, 1930], is strongly connected with its algebraic counterpart: the theory of MV-algebras. Introduced in [Chang, 1958], MV-algebras stand to Łukasiewicz propositional logic as boolean algebras stand to classical logic. We refer to [Cignoli et al., 2000] for a comprehensive study of their general theory and to [Mundici, 2011] for advanced topics. Łukasiewicz logic and MV-algebras are also studied in the general context of t-norm based logics [Hájek, 1998].

MV-algebras are structures $(A, \oplus, ^*, 0)$ of type $(2, 1, 0)$, satisfying some specific identities. The theory of MV-algebras was highlighted by Mundici's categorical equivalence between MV-algebras and Abelian lattice-ordered groups with strong unit (ℓu-groups) [Mundici, 1986]. As a consequence, for any MV-algebra A there exists a unique ℓu-group (G, u) such that A is isomorphic with the unit interval $[0, u]$ of G, endowed with an MV-algebra structure by $x^* = u - x$ and $x \oplus y = (x + y) \wedge u$ for any x, $y \in [0, u]$. Note that the MV-algebraic sum $\oplus$, which can be seen as a non-idempotent disjunction, is the group addition $+$ truncated to the unit interval. Further operations are defined as follows: 1 is 0^*, the Łukasiewicz implication is $x \to y = x^* \oplus y = (u - x + y) \wedge u$ and the Łukasiewicz conjunction is $x \odot y = (x^* \oplus y^*)^* = (x + y - u) \vee 0$ for any x, $y \in [0, u]$. The standard MV-algebra is the real unit interval $[0, 1]$ equipped with the above defined operations. Chang's completeness theorem states that an equation is satisfied in all MV-algebras if and only if it is satisfied in the MV-algebra $[0, 1]$.

Since the real interval $[0, 1]$ is closed to the product operation, a natural problem was to find a complete axiomatization for the variety generated by $([0, 1], \oplus, \cdot, ^*, 0)$, where $([0, 1], \oplus, ^*, 0)$ is the standard MV-algebra and $\cdot : [0, 1] \times [0, 1] \to [0, 1]$ is the product of real numbers. Enriching the structure of MV-algebras, the product was defined as an internal operation for PMV-algebras [Di Nola and Dvurečenskij, 2001] and as a scalar multiplication with scalars from $[0, 1]$ for Riesz MV-algebras [Di Nola and Leuştean, 2014]. In [Lapenta and Leuştean, 2015] the authors defined and studied MV-algebras with both internal product and scalar multiplication under the name of fMV-algebras. The equivalence between MV-algebras and ℓu-groups is generalized for each case, leading to equivalences with particular classes of f-rings, Riesz spaces and f-algebras. Moreover, connections between these structures are proved in [Lapenta and Leuştean, 2016] using the MV-algebraic tensor product defined in [Mundici, 1999].

The logical systems developed for PMV-algebras [Horčík and Cintula, 2004] and Riesz MV-algebras [Di Nola and Leuştean, 2014] are conservative extensions of Łukasiewicz logic. One of the main theorems of Łukasiewicz logic states that, for $n \geq 1$, the term functions with n variables are exactly the continuous $[0, 1]$-valued piecewise linear functions with integer coefficients defined on $[0, 1]^n$ [McNaughton, 1951]. This can be seen as a normal form theorem for Łukasiewicz logic. A similar result was proved in [Di Nola and Leuştean, 2014] for the logical system that has Riesz MV-algebras as models; in this case the piecewise linear functions have real coefficients. In [Montagna and Panti, 2001, Introduction] it is stated that a similar result for PMV-algebras is related to the Pierce-Birkhoff conjecture [Birkhoff and Pierce, 1956] and our aim was to make a deeper investigation of this connection. Consequently, we characterized a subclass of fMV-algebras such that the normal form theorem of the corresponding logical system is a local version of the Pierce-Birkhoff conjecture [Lapenta and Leuştean, 2015].

We overview four equational theories, whose underlying models are MV-algebras, Riesz MV-algebras, PMV-algebras and fMV-algebras. At the core of our presentation lies Theorem 4.1, a general normal form result. The already known normal form theorems for MV-algebras, Riesz MV-algebras and fMV-algebras are straightforward consequences, as well as the normal form theorem for PMV-algebras, which is a new result. In the last section of our paper we emphasize the relation with the Pierce-Birkhoff conjecture.

2 The algebras of Łukasiewicz logic with product

In this section we present the structures of Łukasiewicz logic extended with a product operation that can be either an internal binary one, or a scalar multiplication with scalars from $[0,1]$.

An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$ of type (2,1,0) which satisfies the following properties for any $x, y \in A$:

$$\begin{array}{ll} (A,\oplus,0) \text{ is an Abelian monoid}, & (x^*)^* = x \\ (x^*\oplus y)^*\oplus y = (y^*\oplus x)^*\oplus x, & 0^*\oplus x = 0^* \end{array}$$

We refer to [Cignoli et al., 2000] for all the unexplained notions related to MV-algebras. In any MV-algebra A we can define the following: $1 \stackrel{def}{=} 0^*$, $x \odot y \stackrel{def}{=} (x^* \oplus y^*)^*$, $x \vee y \stackrel{def}{=} (x \odot y^*) \oplus y$ and $x \wedge y \stackrel{def}{=} (x \oplus y^*) \odot y$ for any $x, y \in A$. Hence $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice such that $x \leq y$ if and only if $x \odot y^* = 0$.

A *Riesz MV-algebra* [Di Nola and Leuştean, 2014] is a structure

$$(R, \oplus, ^*, 0, \{r \mid r \in [0,1]\})$$

such that $(R, \oplus, ^*, 0)$ is an MV-algebra and $\{r \mid r \in [0,1]\}$ is a family of unary operation such that the following properties hold for any $x, y \in A$ and $r, q \in [0,1]$:

$$\begin{array}{ll} r(x\odot y^*) = (rx)\odot(ry)^*, & (r\odot q^*)\cdot x = (rx)\odot(qx)^*, \\ r(qx) = (rq)x, & 1x = x. \end{array}$$

A *PMV-algebra*[1] [Di Nola and Dvurečenskij, 2001, Montagna, 2000] is a structure $(P, \oplus, {}^*, \cdot, 0)$ such that $(P, \oplus, {}^*, 0)$ is an MV-algebra, the operation $\cdot : P \times P \to P$ is associative and commutative, and the following identities hold for any $x, y, z \in P$:

$$z \cdot (x \odot y^*) = (z \cdot x) \odot (z \cdot y)^*,$$
$$x \cdot 1 = x.$$

An *fMV-algebra*[2] [Lapenta and Leuştean, 2015] is a structure

$$(A, \oplus, \cdot, {}^*, \{r \mid r \in [0,1]\}, 0)$$

which satisfies the following properties:

$(A, \oplus, \cdot, {}^*, 0)$ is a PMV-algebra,
$(A, \oplus, {}^*, \{r \mid r \in [0,1]\}, 0)$ is a Riesz MV-algebra,
$r(x \cdot y) = (rx) \cdot y = x \cdot (ry)$ for any $x, y, z \in A$ and $\alpha \in [0,1]$.

For $x, y, r \in [0,1]$ we define $x \oplus y = \min\{x+y, 1\}$, $x^* = 1 - x$, while $x \cdot y = xy$ and rx coincide with the product of real numbers. In consequence the real interval $[0,1]$ naturally becomes an MV-algebra, a PMV-algebra, a Riesz MV-algebra and an fMV-algebra. While $[0,1]$, endowed with the appropriate structure, generates the variety of MV-algebras and Riesz MV-algebras, this is no longer true for PMV-algebras and fMV-algebras. This issue will be further discussed in Section 5.

We mention in the sequel one of the most relevant results in the theory of MV-algebras: its connection with the theory of Abelian lattice-ordered groups.

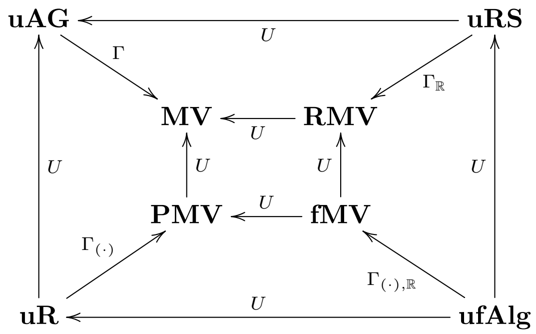
An *ℓu-group* is a pair (G, u), where G is an Abelian lattice-ordered group [Bigard et al., 1977] and u is a strong unit. If (G, u) is an ℓu-group, then $[0, u]_G = ([0, u], \oplus, {}^*, 0)$ is an MV-algebra, where $[0, u] = \{x \in G \mid 0 \le x \le u\}$ and $x \oplus y = u \wedge (x + y)$, $x^* = u - x$ for any $x \in [0, u]$.

If **MV** is the category of MV-algebras with MV-algebra homomorphisms and **uAG** is the category of ℓu-groups equipped with lattice-ordered group homomorphisms that preserve the strong unit, then one defines a functor Γ : $\mathbf{uAG} \to \mathbf{MV}$ by $\Gamma(G, u) = [0, u]_G$ and $\Gamma(h) = h|_{[0,u_1]_{G_1}}$, where (G, u) is an ℓu-group and $h : G_1 \to G_2$ is a morphism in **uAG** between (G_1, u_1) and (G_2, u_2). In [Mundici, 1986], Mundici proved that the functor Γ establishes a categorical equivalence between **uAG** and **MV**.

[1] We assume that PMV-algebras are commutative and unital, while the definition from [Di Nola and Dvurečenskij, 2001] is more general.

[2] As for PMV-algebras, we assume that fMV-algebras are commutative and unital, while the definition from [Lapenta and Leuştean, 2015] is more general.

It is clear that all mentioned algebraic structures are deeply related to each other. In particular, all of them have an MV-algebra reduct. Therefore we can define forgetful functors from the categories **PMV** of PMV-algebras, **RMV** of Riesz MV-algebras and **fMV** of fMV-algebras to **MV**. The categorical equivalence between MV-algebras and ℓu-groups can be generalized for each of this structures to an equivalence with an appropriate class of unital lattice-ordered structures having a lattice-ordered group reduct with a strong unit [Di Nola and Dvurečenskij, 2001, Di Nola and Leuştean, 2014, Lapenta and Leuştean, 2015]. Denoted by **uR** the category of unital f-rings with strong unit, by **uRS** the category of Riesz spaces with strong unit and by **ufAlg** the category of unital f-algebras with strong unit (fu-algebras), we have the following,



3 On the Pierce-Birkhoff conjecture

At the end of the paper [Birkhoff and Pierce, 1956], the authors asked for a characterization of the "*free, commutative, real ℓ-algebra (ℓ-group) with n generators*" and they conjectured that "*it is isomorphic with the l-group of real functions which are continuous and piecewise polynomial of degree at most n over a finite number of pieces*". They asked "*the same problem for the free (commutative) ℓ-rings, for free f-rings*", saying that: *"The former is probably very difficult"*.

Definition 3.1. Let $n \geq 1$ be a natural number.

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *piecewise polynomial* (PWP) function if it is continuous and there is a finite set of polynomials $\{p_1, \ldots, p_k\} \subseteq \mathbb{R}[x_1, \cdots, x_n]$ such that for any $(a_1, \ldots, a_n) \in \mathbb{R}^n$ there exists $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = p_i(a_1, \ldots, a_n)$.
- A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is a *inf-sup-polynomial-definable* (ISD) function if there is a finite set of polynomials $\{q_{ij} | 1 \leq i \leq m, 1 \leq j \leq k\} \subseteq \mathbb{R}[x_1, \cdots, x_n]$ such that $f = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k} q_{ij}$.

We denote by PWP(n) the set of all PWP-functions and by ISD(n) the set of all ISD-functions defined as above. Our notations are inspired by [Delzell, 1989].

The Pierce-Birkhoff conjecture states that

$$PWP(n) = ISD(n) \text{ for any } n \geq 2.$$

In this form, it was formulated by Henriksen and Isbell. The proof for $n \leq 2$ is made é in [Mahé, 1984, 2007], where an unpublished proof of Gus Efroymson is also quoted. In [Mahé, 2007] the author proves that any PWP-function has a representation by inf and sup of polynomials in the whole $\mathbb{R}^3$, except for the union of arbitrary small neighbours of a finite number of points depending only on the function in exams. We refer to [Madden, 2011] for a comprehensive survey on the subject.

Nowadays the conjecture is still inspiring. In [Lucas et al., 2015] the authors say: *"This paper represents a step in our program towards the proof of the Pierce–Birkhoff conjecture."*

Remark 3.1. In Definition 3.1 one may consider piecewise linear functions instead of piecewise polynomial functions. Such functions are called *piecewise homogeneous linear* in [Beynon, 1974], where the following result is proved:

the vector lattice of all piecewise homogeneous linear functions $f : \mathbb{R}^n \to \mathbb{R}$ coincides with the vector lattice of all functions $f : \mathbb{R}^n \to \mathbb{R}$ which can be expressed in the form $\bigvee_{i=1}^{m} \bigwedge_{j=1}^{k} q_{ij}$, where $q_{ij} : \mathbb{R}^n \to \mathbb{R}$ are linear functionals for any i, j.

4 Term functions and piecewise polynomial functions

Let S be a subring of $\mathbb{R}$. Let $\mathcal{L}_0 = \{\oplus, ^*, 0\}$ be the language of MV-algebras, i.e. $\oplus$ is a binary operation, * is unary and 0 is a constant. Let $\mathcal{L}_1 = \mathcal{L}_0 \cup \{\cdot\}$, $\mathcal{L}_{0,S} = \mathcal{L}_0 \cup \{\delta_r \mid r \in [0,1] \cap S\}$, $\mathcal{L}_{1,S} = \mathcal{L}_0 \cup \{\cdot\} \cup \{\delta_r \mid r \in [0,1] \cap S\}$, where $\cdot$ is a binary operation and δ_r is unary operation for any $r \in S \cap [0,1]$. We assume that $\mathcal{L} \in \{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_{0,S}, \mathcal{L}_{1,S}\}$ and that $[0,1]$ is an $\mathcal{L}$-algebra as follows: $x \oplus y = \min\{x+y, 1\}$, $x^* = 1-x$, $x \cdot y = xy$ is the product of real numbers and $\delta_r x = rx$ for any $r \in S$, $x, y \in [0,1]$.

For $n \geq 1$ we define in the usual way the set $Term(n, \mathcal{L})$ of $\mathcal{L}$-terms in n variables (denoted by $X_1, \cdots, X_n$). If $t \in Term(n, \mathcal{L})$ then $\widetilde{t} : [0,1]^n \to$

$[0,1]$ is the term function determined by t, when $[0,1]$ is assumed to have the corresponding $\mathcal{L}$-structure. We further set

$$TF(n,\mathcal{L}) = \{\widetilde{t} : [0,1]^n \to [0,1] \mid t \in Term(n,\mathcal{L})\}.$$

Our aim is to characterize the elements of $TF(n,\mathcal{L})$, by means of piecewise polynomial functions.

Remark 4.1. In order to express better our general results we introduce the notion of *$\mathcal{L}$-polynomial function* as follows:

- an *$\mathcal{L}_0$-polynomial function* is an affine linear function $p : \mathbb{R}^n \to \mathbb{R}$ with integer coefficients;
- an *$\mathcal{L}_1$-polynomial function* is a polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ with integer coefficients;
- an *$\mathcal{L}_{0,S}$-polynomial function* is an affine linear function $p : \mathbb{R}^n \to \mathbb{R}$ with coefficients from S;
- an *$\mathcal{L}_{1,S}$-polynomial function* is a polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ with coefficients from S.

Notation 4.1. In the sequel, the map $\varrho : \mathbb{R} \to [0,1]$ is defined by $\varrho(x) = x \wedge 1 \vee 0$, for any $x \in \mathbb{R}$.

Definition 4.1. Let $n \geq 1$ be a natural number and let S be a subring of $\mathbb{R}$.

- A continuous function $f : [0,1]^n \to [0,1]$ is a *unital piecewise polynomial function with coefficients from S* if there exists a finite set of polynomials, called *components*, $\{p_i : \mathbb{R}^n \to \mathbb{R} \mid 1 \leq i \leq k\} \subseteq S[x_1,\ldots,x_n]$ such that for any $(a_1,\ldots,a_n) \in \mathbb{R}^n$ there exists $i \in \{1,\ldots,k\}$ with $f(a_1,\ldots,a_n) = p_i(a_1,\ldots,a_n)$. We denote by $PWP(n,\mathcal{L})$ the set of all such functions whose components are $\mathcal{L}$-polynomial functions.
- A continuous function $f : [0,1]^n \to [0,1]$ is a *unital inf-sup-definable function with coefficients from S* if there exists a finite set of polynomials, called *components*, $\{q_{ij} : \mathbb{R}^n \to \mathbb{R} | 1 \leq i \leq m, 1 \leq j \leq k\} \subseteq S[x_1,\ldots,x_n]$ such that $f = \bigvee_{i=1}^m \bigwedge_{j=1}^k (\varrho \circ q_{ij})$. We denote by $ISD(n,\mathcal{L})$ the set of all such functions whose components are $\mathcal{L}$-polynomial functions.

The following result generalizes [McNaughton, 1951, Theorem 2], [Cignoli et al., 2000, Proposition 3.1.8], [Di Nola and Leuştean, 2014, Theorem 10], [Lapenta and Leuştean, 2015, Proposition 3.5]. We give the proof for the sake of completeness.

Proposition 4.1. $TF(n,\mathcal{L}) \subseteq PWP(n,\mathcal{L})$.

Proof. Let t be a term in $Term(n,\mathcal{L})$. The result will be proved by structural induction on t.
- If $t = X_i$ for some $i \leq n$, then $\widetilde{t} = \pi_i^n$ and it trivially belongs to $PWP(n,\mathcal{L})$.
- If $t = t_1^*$, then $\widetilde{t} = (\widetilde{t_1})^*$. By induction hypothesis there exists an integer h and some polynomials $q_1,\ldots,q_h \in S[x_1,\ldots,x_n]$ such that for any point in the n-cube, $\widetilde{t_1}$ coincide with one of them. Then $1-q_1,\ldots,1-q_h$ are the components of $\widetilde{t}$.
- If $t = t_1 \oplus t_2$, let $q_1,\ldots, q_m$ be the components of $\widetilde{t_1}$ and $p_1,\ldots,p_k$ be the components of $\widetilde{t_2}$. Then $\widetilde{t}$ is defined by the polynomials $\{1\} \cup \{s_{ij}\}_{i,j}$, where $s_{ij} = 1 - q_i + p_j$ for any $i \in \{1,\ldots,s\}$ and $j \in \{1,\ldots,r\}$.
- If t is $\delta_r(t_1)$ for some $r \in [0,1] \cap S$ and $q_1,\ldots,q_s$ are the components of $\widetilde{t_1}$, then $rq_1,\ldots,rq_s$ are the components of $\widetilde{t}$.
- If $t = t_1 \cdot t_2$, let $q_1,\ldots, q_m$ be the components of $\widetilde{t_1}$ and $p_1,\ldots,p_k$ be the components of $\widetilde{t_2}$. Then $\widetilde{t}$ is defined by the polynomials $q_i \cdot p_j$, for any $i \in \{1,\ldots,s\}$ and $j \in \{1,\ldots,r\}$. □

Lemma 4.1. *[Di Nola and Leuştean, 2014, Lemma 9] For any x, $y \in \mathbb{R}$ the following properties hold:*

(a) $(x \vee 0) + (y \vee 0) \geq (x+y) \vee 0$,

(b) $x \geq 0$ *iff* $\varrho(-x) = 0$,

(c) $\varrho(x) = \varrho(x \vee 0)$.

Lemma 4.2. *[Di Nola and Leuştean, 2014, Lemma 10] If $g : [0,1]^n \to \mathbb{R}$ and $h : [0,1]^n \to [0,1]$ then the following properties hold.*

(a) $\varrho \circ (g+h) = ((\varrho \circ g) \oplus h) \odot (\varrho \circ (g+1))$.

(b) $\varrho \circ (1-g) = 1 - (\varrho \circ g)$.

The following result is a generalization of [McNaughton, 1951, Theorem 1], [Cignoli et al., 2000, Lemma 3.1.9], [Di Nola and Leuştean, 2014, Proposition 6], [Lapenta and Leuştean, 2015, Proposition 4.3]. In spite of the fact that the proof is very similar to the one in [Lapenta and Leuştean, 2015], it covers at least one additional relevant case: for $\mathcal{L} = \mathcal{L}_1$, this result and Theorem 4.1 lead to a better understanding of the free PMV-algebra with n free generators.

We note that, following the proof, one can extract an algorithm that generates the term associated to a given $\mathcal{L}$-polynomial function (see Figure 1).

Proposition 4.2. *Let S be a subring of $\mathbb{R}$.*
(a)For any $\mathcal{L}$-polynomial function $p : [0,1]^n \to \mathbb{R}$ there exists $t \in Term(n,\mathcal{L})$ such that $\varrho \circ p = \widetilde{t}$.
(b)For any $g \in ISD(n,\mathcal{L})$ there exists $t \in Term(n,\mathcal{L})$ such that $g = \widetilde{t}$.

Proof. We give the proof for $\mathcal{L} = \mathcal{L}_{1,S}$, which is the most general one.

(a) Let $p : [0,1]^n \to \mathbb{R}$ be an polynomial function. Let k be the degree of p. Then

$$p(x_1, \ldots, x_n) = \sum_{i_1+\ldots+i_n \leq k} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $c_{i_1,\ldots,i_n} \in S$ for any choice of the indexes. Since any $c_{i_1,\ldots,i_n}$ can be written as a sum of a finite number of elements in $[-1,1] \cap S$, we assume that

$$p(x_1, \ldots, x_n) = r_m y_m + \cdots + r_{p+1} y_{p+1} + r_p + \cdots + r_1$$

where $m \geq 1$ and $p \geq 0$ are natural numbers, $p \leq m$, $r_j \in ([-1,1] \cap S) \setminus \{0\}$ for any $j \in \{1, \cdots, m\}$ and $y_j \in \{x_1^{i_1} \cdots x_n^{i_n} \mid i_1 + \cdots + i_n \leq k\}$ for any $j \in \{p+1, \cdots, m\}$.

We prove the theorem by induction on $m \geq 1$. In the sequel we denote by $\mathbf{x}$ an element $(x_1, \ldots, x_n)$ from $[0,1]^n$.

Initial step $m = 1$. We have $p(\mathbf{x}) = r$ for any $\mathbf{x} \in [0,1]^n$ or $p(\mathbf{x}) = r x_1^{i_1} \cdots x_n^{i_n}$ for any $\mathbf{x} \in [0,1]^n$ where $r \in ([-1,1] \cap S) \setminus \{0\}$ and $\{i_1, \ldots, i_n\}$ is a suitable set of index.

If $r \in [-1,0) \cap S$ then $\varrho \circ p = 0$ so $\varrho \circ p = \widetilde{t}$ for $t = 0$.

If $r \in (0,1] \cap S$ then $p = \varrho \circ p$. It follows that $p = \widetilde{t}$ where $t = \delta_r(0^*)$ if $p(\mathbf{x}) = r$ for any $\mathbf{x} \in [0,1]^n$ and $t = \delta_r(X_1^{i_1} \cdots X_n^{i_n})$ if $p(\mathbf{x}) = r x_1^{i_1} \cdots x_n^{i_n}$ for any $\mathbf{x} \in [0,1]^n$.

Induction step. We take $p = g + h$ where $\varrho \circ g = \widetilde{t_1}$ for some term t_1 and there is $r \in ([-1,1] \cap S) \setminus \{0\}$ and a suitable choice of index for $i_1, \ldots, i_n$ such that $h(\mathbf{x}) = r$ for any $\mathbf{x} \in [0,1]^n$, or $h(\mathbf{x}) = r x_1^{i_1} \cdots x_n^{i_n}$ for any $\mathbf{x} \in [0,1]^n$. We consider two cases.

Case 1. If $r \in (0,1]$ then $h : [0,1]^n \to [0,1]$ so $\varrho \circ p = ((\varrho \circ g) \oplus h) \odot (\varrho \circ (1+g))$ by Lemma 4.2 (a). Following the initial step, there is a term t_2 such that $h = \widetilde{t_2}$. We notice that $1 + g = 1 - (-g)$ and the induction hypothesis holds for $(-g)$, then there is a term t_3 such that $\varrho \circ (-g) = \widetilde{t_3}$. It follows by Lemma 4.2 (b) that $\varrho \circ (1+g) = 1 - \widetilde{t_3} = \widetilde{t_3}^*$, and $\varrho \circ p = \widetilde{t}$ where $t = (t_1 \oplus t_2) \odot t_3^*$.

Case 2. If $r \in [-1,0)$, then $g + h = (g-1) + (1+h)$ and $1 + h : [0,1]^n \to [0,1]$. By Lemma 4.2 (a) we get

$$\varrho \circ p = ((\varrho \circ (g-1)) \oplus (1+h)) \odot (\varrho \circ g).$$

Following the initial step, there is a term t_2 such that $-h = \widetilde{t_2}$, so $1 + h = 1 - (-h) = \widetilde{t_2}^*$. In the sequel we have to find a term t_3 that corresponds to $\varrho \circ (g-1)$, where

$$g(\mathbf{x}) = r_m y_m + \cdots + r_{p+1} y_{p+1} + r_p + \cdots + r_1$$

with $r_j \in ([-1,1] \cap S) \setminus \{0\}$ for any $j \in \{1, \cdots, m\}$ and y_j in $\{x_1^{i_1} \cdots x_n^{i_n} \mid i_1 + \cdots + i_n \leq k\}$ for any $j \in \{p+1, \cdots, m\}$.

Case 2.1. If $r_j \leq 0$ for any $j \in \{1, \cdots, m\}$ then $g - 1 \leq 0$, so $\varrho \circ (g-1) = 0 = \widetilde{t_3}$ with $t_3 = 0$.

Case 2.2. If there is $j_0 \in \{1, \cdots, p\}$ such that $r_{j_0} > 0$, then

$$(g-1)(\mathbf{x}) = r_m y_m + \cdots + r_{p+1} y_{p+1} + r_p + \cdots + (r_{j_0} - 1) + \cdots + r_1$$

and $r_{j_0} - 1 \in [-1, 0)$, so the induction hypothesis applies to $g - 1$. Then there exists a term t_3 such that $\varrho \circ (g-1) = \widetilde{t_3}$.

Case 2.3. If there is $j_0 \in \{p+1, \cdots, m\}$ such that $r_{j_0} > 0$, then we set $h_0(\mathbf{x}) = r_{j_0} y_{j_0}$ and

$$g_0(\mathbf{x}) = g(\mathbf{x}) - r_{j_0} y_{j_0} - 1.$$

It follows that $g - 1 = g_0 + h_0$ such that g_0 satisfies the induction hypothesis and $h_0 : [0,1]^n \to [0,1]$. We are in the hypothesis of *Case 1*, so there exists a term t_3 such that $\varrho \circ (g-1) = \widetilde{t_3}$.

Summing up, we get $\varrho \circ (g+h) = \widetilde{t}$ with $t = ((t_2 \oplus t_3^*) \odot t_1)$.

(b) This is straightforward by (a): for any g_{ij} there exist a term $t_{ij} \in Term_n(S)$ such that $\varrho \circ g_{ij} = \widetilde{t_{ij}}$.

Then

$$g = \varrho \circ g = \bigvee_{i \in I} \bigwedge_{j \in J} \varrho \circ g_{ij} = \bigvee_{i \in I} \bigwedge_{j \in J} \widetilde{t_{ij}}.$$

If $t = \bigvee_{i \in I} \bigwedge_{j \in J} t_{ij}$, we get $g = \widetilde{t}$. □

Remark 4.2. If $f : [0,1]^n \to \mathbb{R}$ is a polynomial function with real coefficients then we represent $f : [0,1]^n \to \mathbb{R}$ as

$$f(x_1, \ldots, x_n) = r_m y_m + \cdots + r_{p+1} y_{p+1} + r_p + \cdots + r_1$$

where $p \leq m$, $r_j \in [-1,1] \setminus \{0\}$, and $y_j = x_1^{i_j^1} \cdots x_n^{i_j^n}$ is a monomial for any $j > p$. Using this representation one can easily extract an algorithm from the proof of Theorem 4.1, which returns an $\mathcal{L}$-term corresponding to f. In Figure 1 we describe the algorithm for $\mathcal{L} = \mathcal{L}_{1,S}$.

We note that the polynomial f is represented by

$$(p, r_1, \ldots, r_p, (r_{p+1}, \mathbf{i}_{p+1}), \ldots, (r_m, \mathbf{i}_m)),$$

where $\mathbf{i}_j = (i_j^1, \ldots, i_j^n)$ for any $j \in \{p+1, \ldots, m\}$. In particular cases the representation can be simplified as follows:

- for $\mathcal{L} \in \{\mathcal{L}_0, \mathcal{L}_{0,S}\}$ the monomials y_j satisfy the condition $i_j^1 + \cdots + i_j^n = 1$, so y_j are of the form x_k;

- for $\mathcal{L} \in \{\mathcal{L}_0, \mathcal{L}_1\}$ and for any j we have $r_j \in \{-1, 1\}$.

The algorithm for $\mathcal{L} = \mathcal{L}_{0,\mathbb{R}}$ is described in [Gerla et al., 2013].

```
// we use the notation i_k = (i_k^1, ..., i_k^n)
function Term(p, r_1, ..., r_p, (r_{p+1}, i_{p+1}), ... (r_m, i_m))
{
(F1) if r_k ≤ 0 for any k ∈ {1, ..., m} then return(0);
(F2) find k ∈ {1, ..., m} such that r_k > 0;
     if k ≤ p then ψ := δ_{r_k}(0*) else ψ := δ_{r_k}(X_1^{i_k^1} ··· X_n^{i_k^n});
(F3) if m = 1 then return(ψ);
(F4) if k ≤ p then
{φ = Term(p − 1, r_1, ..., r_{k−1}, r_{k+1}, ... r_p, (r_{p+1}, i_{p+1}), ..., (r_m, i_m));
χ =Term(p − 1,−r_1, ...,−r_{k−1}, −r_{k+1}, ...− r_p, (−r_{p+1}, i_{p+1}), ..., (−r_m, i_m))
} else
{φ = Term(p, r_1, ..., r_p, ..., (r_{k−1}, i_{k−1})(r_{k+1}, i_{k+1}), ..., (r_m, i_m));
χ = Term(p, −r_1, ..., −r_p, ..., (−r_{k−1}, i_{k−1}), (−r_{k+1}, i_{k+1}), ..., (−r_m, i_m))
}; return((φ ⊕ ψ) ⊙ ¬χ)
}
```

Figure 1: The term corresponding to a polynomial function

The following theorem summarizes our results.

Theorem 4.1. *$ISD(n, \mathcal{L}) \subseteq TF(n, \mathcal{L}) \subseteq PWP(n, \mathcal{L})$ for $\mathcal{L} \in \{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_{0,S}, \mathcal{L}_{1,S}\}$.*

5 Normal form theorems and the Pierce-Birkhoff conjecture

Let S be a subring of $\mathbb{R}$ and $\mathcal{L}$ as above. In the light of Theorem 4.1, one may ask if the following equalities hold:

$$(*) \qquad ISD(n, \mathcal{L}) = TF(n, \mathcal{L}) = PWP(n, \mathcal{L}).$$

This problem has the same flavor as the Pierce-Birkhoff conjecture. Moreover, for $\mathcal{L} = \mathcal{L}_0$ this is the McNaughton theorem [McNaughton, 1951, Mundici, 1994]. In the following we discuss the property $(*)$ from logical perspective.

If $\mathcal{L} \in \{\mathcal{L}_1, \mathcal{L}_{0,S}, \mathcal{L}_{1,S}\}$ and $[0, 1]_{\mathcal{L}}$ is an $\mathcal{L}$-algebra defined in the previous section, then $TF(n, \mathcal{L})$ is an $\mathcal{L}$-algebra with pointwise operations. By general results in universal algebra $TF(n, \mathcal{L})$ is the free $\mathcal{L}$-structure in the variety generated by $[0, 1]_{\mathcal{L}}$.

We already mentioned that $\mathcal{L}_0$ is the language of Łukasiewicz logic. One can easily see that $\mathcal{L}_1$, $\mathcal{L}_{0,\mathbb{R}}$, $\mathcal{L}_{1,\mathbb{R}}$ are, respectively, the language of PMV-algebras, Riesz MV-algebras and fMV-algebras.

We further note that $[0,1]_{\mathcal{L}_0}$ generates the variety of MV-algebras [Chang, 1959, Cignoli et al., 2000] and $[0,1]_{\mathcal{L}_{0,\mathbb{R}}}$ generates the variety of Riesz MV-algebras [Di Nola and Leuştean, 2014]. Let $\mathcal{L}\mathbb{R}$ be the propositional calculus which has Riesz MV-algebras as models. In this context $(*)$ holds [Di Nola and Leuştean, 2014] and it can be seen as a local version of the result from Remark 3.1.

The situation is different for PMV-algebras, since the standard model $[0,1]_{\mathcal{L}_1}$ generates only a proper subvariety [Horčík and Cintula, 2004]. Moreover, due to a result of Isbell [Isbell, 1972] this variety is not finitely axiomatizable. Montagna proved that the proper quasi-variety of PMV$^+$-algebras defined by the quasi-identity:

$$x \cdot x = 0 \Rightarrow x = 0 \text{ for any } x$$

is generated, as a quasi-variety, by $[0,1]_{\mathcal{L}_1}$ [Montagna, 2005]. Consequently, the equality $(*)$ for $\mathcal{L}_1$ is a normal form theorem for the system PL', described in [Horčík and Cintula, 2004], that has PMV$^+$-algebras as models. A similar analysis is made in [Lapenta and Leuştean, 2015] for fMV-algebras; in this case the quasi-variety generated by the standard model $[0,1]_{\mathcal{L}_{1,\mathbb{R}}}$ is the class of FR$^+$-algebras and the corresponding propositional calculus is denoted $\mathcal{FMVL}^+$.

Finally we note that all these logical systems are conservative extensions of Łukasiewicz ∞-valued logic and the $\mathcal{L}$-structure $TF(n,\mathcal{L})$ is, up to isomorphism, the Lindenbaum-Tarski algebra of the corresponding propositional calculus. The property $(*)$ is a normal form theorem and we summarize below the results known so far.

$\mathcal{L}$	Logic	Algebra	$ISD(n,\mathcal{L})= PWP(n,\mathcal{L})$
$\mathcal{L}_0$	Łuk	MV-algebras	true [McNaughton, 1951]
$\mathcal{L}_{0,\mathbb{R}}$	$\mathcal{L}\mathbb{R}$	Riesz MV-algebras	true [Di Nola and Leuştean, 2014]
$\mathcal{L}_1$	PL'	PMV$^+$-algebras	open problem
$\mathcal{L}_{1,\mathbb{R}}$	$\mathcal{FMVL}^+$	FR$^+$-algebras	true if $n \leq 2$ open problem if $n > 2$ [Lapenta and Leuştean, 2015]

In the following we emphasize the results for the system $\mathcal{FMVL}^+$. Note that in this case the unital piecewise polynomial functions from Definition 4.1 have the components as the continuous piecewise polynomial functions from the Pierce-Birkhoff conjecture.

If $f : \mathbb{R}^n \to [0,1]$ is a continuous piecewise polynomial function then $f|_{[0,1]^n}$ is a continuous piecewise polynomial function defined on the unit cube. In general it is not known if any piecewise polynomial function defined on $[0,1]^n$ is the restriction of a continuous piecewise polynomial function defined on $\mathbb{R}^n$, but for $n = 2$ a positive answer is given in [Fischer and Marshall, 2013]. As a direct consequence and using Mahé's proof for the Pierce-Birkhoff conjecture [Mahé, 1984], for $n \leq 2$ we get

$$ISD(n, \mathcal{L}_{1,\mathbb{R}}) = TF(n, \mathcal{L}_{1,\mathbb{R}}) = PWP(n, \mathcal{L}_{1,\mathbb{R}}).$$

The above result is a normal form theorem for the propositional calculus $\mathcal{FMVL}^+$ and describes the functions corresponding to the formulas in two variables. The following result can be interpreted as a local version of the Pierce-Birkhoff theorem.

Conjecture 5.1. *[Lapenta and Leuştean, 2015] For $n > 2$,*

$$ISD(n, \mathcal{L}_{1,\mathbb{R}}) = TF(n, \mathcal{L}_{1,\mathbb{R}}) = PWP(n, \mathcal{L}_{1,\mathbb{R}}).$$

We note that the above result does not immediately imply and it is not immediately implied by the original Pierce-Birkhoff conjecture and it might be equally hard to prove.

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