On the semisimple tensor product of MV-algebras

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Abstract

In this note we show how the MV-algebraic tensor product can be used to define natural adjunctions between the algebraic structures of Łukasiewicz logic with product. In order to do this, we define the semisimple tensor PMV-algebra of a semisimple MV-algebra.

Keywords: MV-algebras, tensor product, scalar extension property, tensor algebra, amalgamation property, direct limits.

Introduction

Mathematical fuzzy logic can be thought as a way to approach Zadeh's fuzzy set theory in a *narrow* sense, giving a formal deductive system in which one deals with vagueness in a rigorous mathematical way. Within the various logics that can be studied from this perspective, a privileged role is played by Łukasiewicz logic. This logic was introduced by Jan Łukasiewicz one century ago and it had an important breakthrough in the late Fifties, when C.C. Chang developed an algebraic semantics for it, by introducing MV-algebras.

The theory of MV-algebras has been intensively studied since then, and among all subclasses of MV-algebras this paper focuses on the semisimple ones. These are subalgebras of the algebra of all continuous $[0,1]$-valued functions defined on some appropriate compact Hausdoff space. From a different perspective, semisimple MV-algebras are the ones isomorphic to some *bold algebra* of fuzzy sets, that is, an algebra of fuzzy sets closed under Łukasiewicz operations.

The notable growth of the theory of MV-algebras was driven by the categorical equivalence between MV-algebras and Abelian lattice-ordered groups with a strong unit, proved by D. Mundici in [20]. Consequently, a natural hierarchy of structures has arisen from MV-algebras, along the same line that connects groups, rings, vector spaces and algebras (where algebras are intended in the sense of commutative algebra, that is, as structures with both a scalar

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product and a ring product). Starting from the Nineties, expansions of MV-algebras have been defined: one can find the notions of PMV-algebras [8, 19] (MV-algebras endowed with a ring product), DMV-algebras and Riesz MV-algebras [9, 10, 17] (MV-algebras endowed with a scalar multiplication), fMV-algebras [15] (MV-algebras endowed with both a ring product and a scalar product). These structures and the corresponding logical systems have been studied, starting from their categorical equivalences with lattice-ordered rings (for PMV-algebras), vector lattices (for Riesz MV-algebras and DMV-algebras), f-algebras (for fMV-algebras.

In this paper we define a universal construction that allows to endow an MV-algebra with a ring-like product, obtaining a PMV-algebra. We define and investigate this construction in Section 2, calling it the *tensor PMV-algebra* of a semimple MV-algebra. In Section 3 we lift the results to a categorical level, obtaining two pairs of adjoint functors, connecting MV-algebras with PMV-algebras and Riesz MV-algebras with fMV-algebras. Using our construction, in Section 2.1 we prove the amalgamation property for semisimple PMV-algebras, semisimple Riesz MV-algebras and semisimple fMV-algebras, and we characterize free objects in each category. Finally, in Section 3.1 we transfer all results to lattice-ordered structures via categorical equivalence.

1. Preliminaries

MV-algebras are the variety of lattice-ordered algebras generated by the unit interval $[0,1]$ endowed with a binary operation $x \oplus y = \min(x+y,1)$, an involutive negation $x^* = 1-x$, a top element 1, and a bottom element 0. If one considers MV-algebras enriched with a product operation, several approaches appear in literature, since the product can be a binary internal operation, a scalar multiplication or a combination of both [8, 19, 14, 10]. To give formal definitions in a concise manner, let us consider the notion of bilinear function. Let A, B, C be MV-algebras, a function $\omega : A \to B$ is called *linear* if $\omega(x \oplus y) = \omega(x) \oplus \omega(y)$ whenever $x \leq y^*$. A function $\beta : A \times B \to C$ is *bilinear* if $\beta(-, y)$ and $\beta(x, -)$ are linear for any $x \in A$ and $y \in B$. These notions allow us to define MV-algebras with product in a uniform way, as described in Table 1. We remark that a more general definition of PMV-algebras and fMV-algebras can be found in [8, 15] respectively, but here we focus our attention on unital and commutative structures.

The present investigation is centred on the class of *semisimple* MV-algebras. Such algebras enjoy a crucial functional representation. Indeed, any semisimple MV-algebra is isomorphic to an MV-subalgebra of $[0,1]$-valued continuous functions defined over some compact Hausdorff space [7], namely the space of maximal ideals of the algebra. In symbols, $A \lesssim C(Max(A)) \subseteq [0,1]^{Max(A)}$. A unital PMV-algebra (Riesz MV-algebra or unital fMV-algebra) is semisimple if its MV-algebra reduct is semisimple.

If we look at MV-algebras as an algebraic category, it is possible to obtain an equivalence with the category of Abelian lattice-ordered groups with strong unit [20], *ℓu-groups* for short. The equivalence is given by the functor

Structure	Definition
$(P, \oplus, \cdot, ^*, 0)$ unital and commutative PMV algebra [8, 19]	$(P, \oplus, ^*, 0)$ MV-algebra $\cdot : P \times P \to P$ bilinear, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ and $x \cdot 1 = 1 \cdot x = x$
$(R, \oplus, ^*, \{\alpha \mid \alpha \in [0,1]\}, 0)$ Riesz MV- algebra [10]	$(R, \oplus, ^*, 0)$ MV-algebra $(\alpha, x) \mapsto \alpha x$ bilinear, $(\alpha \cdot \beta)x = \alpha(\beta x)$ and $1x = x$
$(A, \oplus, \cdot, ^*, \{\alpha \mid \alpha \in [0,1]\}, 0)$ unital and commutative fMV- algebra [15]	$(A, \oplus, \cdot, ^*, 0)$ unital PMV-algebra $(A, \oplus, ^*, \{\alpha \mid \alpha \in [0,1]\}, 0)$ R. MV-algebra $\alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y)$

Table 1: Algebraic hierarchy.

$\Gamma : \mathbf{auG} \to \mathbf{MV}$ defined on objects by $\Gamma(G, u) = [0, u]_G = \{x \in G \mid 0 \leq x \leq u\}$. Moreover, through Γ, semisimple MV-algebras correspond to Archimedean ℓu-groups, that is, groups in which $na \leq b$ for any pair of elements a, b and any $n \in \mathbb{N}$ implies that $a \leq 0$ [2]. We shall denote by $\mathbf{MV_{ss}}$ the full subcategory of semisimple MV-algebras and by $\mathbf{auG_a}$ the full subcategory of Archimedean ℓu-groups. Extending Γ, similar equivalences are proved for PMV-algebras, Riesz MV-algebras and fMV-algebras. The functors that establish the equivalences are denoted by $\Gamma_{(\cdot)}$, $\Gamma_{\mathbb{R}}$ and Γ_f respectively. See [2] for details on the above mentioned structures and [8, 10, 15] for details on the categorical equivalences. In Table 2, notations are set for the categories of semisimple MV-algebras and the equivalent categories of Archimedean ℓu-groups that will be used subsequently.

Category	Objects
$\mathbf{uPMV_{ss}}$ $\mathbf{uR_a}$	unital and semisimple PMV-algebras, unital and Archimedean ℓ-rings with strong unit,
$\mathbf{RMV_{ss}}$ $\mathbf{uRS_a}$	semisimple Riesz MV-algebras, Archimedean Riesz spaces with strong unit,
$\mathbf{ufMV_{ss}}$ $\mathbf{fuAlg_a}$	unital and semisimple fMV-algebras, unital and Archimedean f-algebras with strong unit.

Table 2: Categories of MV-algebras and related ℓ-structures.

We remark that in the case of ℓu-rings or fu-algebras, the unit interval has to be closed under the ring product, that is, it is required that $u \cdot u \leq u$. Moreover, there are natural forgetful functors between the above-defined categories, which commute with Γ and its generalizations. Consequently, a natural problem is to define appropriate left adjoints for these forgetful functors. We started this investigation in [16], where the key tool was the *semisimple tensor product of MV-algebras* [21].

Given two semisimple MV-algebras A and B, their (semisimple) tensor product is the semisimple MV-algebra $A \otimes B$ together with a universal bimorphism $\beta_{A,B} : A \times B \to A \otimes B$. Note that a *bimorphism* is a bilinear function that is $\vee$-preserving and $\wedge$-preserving in each component. The universal property satisfied by $\beta_{A,B}$ is the following: for any semisimple MV-algebra C and for any bimorphism $\beta\colon A \times B \to C$, there is a unique homomorphism of MV-algebras $\omega\colon A \otimes B \to C$ such that $\omega \circ \beta_{A,B} = \beta$. For $a \in A$ and $b \in B$ we denote by $a \otimes b$ the element $\beta_{A,B}(a, b)$. As expected, $A \otimes B$ is generated by $\beta_{A,B}(A \times B)$. Recalling that semisimple MV-algebras are isomorphic to subalgebras of continuous functions, we get a functional representation for the tensor product: if A and B are semisimple MV-algebras, X denotes $Max(A)$ and Y denotes $Max(B)$,

$$A \otimes B = \langle a \cdot b \mid a \in A \subseteq C(X),\ b \in B \subseteq C(Y)\rangle_{MV} \subseteq C(X \times Y), \qquad (1)$$

where $a \cdot b$ is the usual product between functions.

Finally, in [16] it is proved that the semisimple tensor product provides an adjoint to the forgetful functor $\mathcal{U}_{\mathbb{R}}\colon \mathbf{RMV_{ss}} \to \mathbf{MV_{ss}}$. The adjoint is the functor $\mathcal{T}_{\otimes} : \mathbf{MV_{ss}} \to \mathbf{RMV_{ss}}$, defined by $\mathcal{T}_{\otimes}(B) = [0, 1] \otimes B$ on objects and via the following universal property on arrows: for any homomorphism of MV-algebras $f : A \to B$, $\mathcal{T}_{\otimes}(f)$ is the unique homomorphism of Riesz MV-algebras such that $\mathcal{T}_{\otimes}(f)(1 \otimes a) = 1 \otimes f(a)$, for any $a \in A$, which exists by [16, Corollary 3.1]. Analogously, $\mathcal{F}_{\otimes} : \mathbf{uPMV_{ss}} \to \mathbf{ufMV_{ss}}$ is defined on objects by $\mathcal{F}_{\otimes}(P) = [0, 1] \otimes P$ and on arrows by the same universal property of $\mathcal{T}_{\otimes}$. The functor $\mathcal{F}_{\otimes}$ provides an adjoint to the forgetful functor from $\mathbf{ufMV_{ss}}$ to $\mathbf{uPMV_{ss}}$.

Remark 1.1. The tensor product of MV-algebras has been defined in the non-semisimple case as well, although it does not enjoy a functional representation similar to the one of Equation (1). For this reason, it is still an open problem whether the above-mentioned adjunctions can be generalized to the non-semisimple case.

2. The semisimple tensor PMV-algebra of a semisimple MV-algebra

In this section we provide the algebraic tools needed to obtain the missing adjunctions between $\mathbf{MV_{ss}}$ and $\mathbf{uPMV_{ss}}$, and between $\mathbf{RMV_{ss}}$ and $\mathbf{ufMV_{ss}}$. The key ingredient will be an "MV-algebraic" version of the classical construction of the *tensor algebra* of a vector space. We start by proving that the semisimple tensor product of MV-algebras is associative. We recall once again that any semisimple MV-algebra is isomorphic to a subalgebra of $C(X)$, where X is the compact Hausdorff space $Max(A)$.

Proposition 2.1. *The semisimple tensor product of MV-algebras is associative. That is,* $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ *and it coincides with the MV-algebra* $\langle a \cdot b \cdot c \mid a \in A, b \in B, c \in C\rangle \subseteq C(X \times Y \times Z)$, *where* $X = Max(A)$, $Y = Max(B)$ *and* $Z = Max(C)$.

Proof. Let M be the MV-subalgebra of $C(X \times Y \times Z)$ generated by $a \cdot b \cdot c$, where $\cdot$ is the usual product between functions. By Equation (1),

$$(A \otimes B) \otimes C = \langle f \cdot c \mid f \in A \otimes B, c \in C \rangle_{MV},$$

the MV-algebra generated by the product of $f \in A \otimes B$ and $c \in C$. We want to prove that $\langle f \cdot c \mid f \in A \otimes B, c \in C \rangle_{MV} = M$.

Trivially $M \subseteq \langle f \cdot c \mid f \in A \otimes B, c \in C \rangle_{MV}$. We prove the other inclusion by induction on the construction of $f \in A \otimes B$.

(i) If $f = a \otimes b = a \cdot b$, it is trivial: $(a \cdot b) \cdot c = a \cdot b \cdot c \in M$.

(ii) Let f be an element of $A \otimes B$ such that $f \cdot c \in M$. Then $f^* \cdot c = (\mathbf{1} - f) \cdot c = c - f \cdot c \in M$, using the induction hypothesis and the fact that $\mathbf{0} \leq c - f \cdot c \leq \mathbf{1}$ in $C(X \times Y \times Z)$.

(iii) Let $f = f_1 \oplus f_2$ be an element of $A \otimes B$ such that $f_1 \cdot c$ and $f_2 \cdot c$ belong to M. Since we deal with subalgebras of continuous functions, it is easily seen that the product distributes over $\wedge$, and therefore we have

$$f \cdot c = (f_1 \oplus f_2) \cdot c = (f_1 + (f_1^* \wedge f_2)) \cdot c = f_1 \cdot c + (f_1^* \cdot c \wedge f_2 \cdot c),$$

which belongs to M by induction hypothesis and (ii).

Therefore $M = (A \otimes B) \otimes C$. In the same way we prove that $M = A \otimes (B \otimes C)$, settling the claim. □

We are ready to give the main definition of this paper. Let A be a semisimple MV-algebra and let X be $Max(A)$. We define:

$$T^1(A) = A, \quad T^n(A) = T^{n-1}(A) \otimes A,$$

where $\otimes$ is the semisimple tensor product. By Proposition 2.1,

$$T^n(A) = \langle f_1 \cdot \ldots \cdot f_n \mid f_i \in A, i = 1 \ldots n \rangle_{MV} \subseteq C(X^n),$$

and $\overbrace{1 \otimes \ldots \otimes 1}^{n} = \overbrace{1 \cdot \ldots \cdot 1}^{n}$ is the top element of $T^n(A)$ for every n.
For any $n, m \in \mathbb{N}$ with $n \leq m$ we define

$\epsilon_{n,n}$ as the identity homomorphism on $T^n(A)$;

$\epsilon_{n,m} : T^n(A) \to T^m(A)$, by $\mathbf{x} \mapsto \mathbf{x} \otimes (1 \otimes \ldots \otimes 1)$,

where by associativity $T^m(A) \simeq T^n(A) \otimes T^{m-n}(A)$. By [16, Proposition 2.1], $\epsilon_{n,m}$ is the embedding in the semisimple tensor product and $\epsilon_{m,k} \circ \epsilon_{n,m} = \epsilon_{n,k}$. Each $T^n(A)$ is semisimple by construction, and $(T^n(A), \epsilon_{n,m})$ is a direct system of semisimple MV-algebras. Consider now the disjoint union

$$\bigsqcup_{n \in \mathbb{N}} T^n(A),$$

and define an equivalence relation on it by

$$(x,n) \sim (y,m) \ \Leftrightarrow \ \text{there exists } k \geq n,m \text{ such that } \epsilon_{n,k}(x) = \epsilon_{m,k}(y).$$

The quotient MV-algebra $T(A) = \bigsqcup_{n\in\mathbb{N}} T^n(A)/_{\sim}$ is the direct limit of the direct system, and $\epsilon_{n,A} : T^n(A) \to T(A)$ is the canonical morphism that maps each element in its equivalence class. When there is no danger of confusion, we will denote $\epsilon_{n,A}$ by ϵ_n.

Definition 2.2. We call $T(A)$ the *Tensor PMV-algebra* of the MV-algebra A.

Remark 2.3. It is well known from general properties of direct limits (see e.g. [13, § 21]) that $\epsilon_m \circ \epsilon_{n,m} = \epsilon_n$ for any $n \leq m$, that the diagram in Figure 1 is commutative, and that any map $\epsilon_n : T^n(A) \to T(A)$ is an embedding.

$$\begin{tikzcd}
T^n(A) \arrow[r, "\epsilon_{n,m}"] \arrow[dr, "\epsilon_n"'] & T^m(A) \arrow[r, "\epsilon_{m,k}"] \arrow[d, "\epsilon_m"'] & T^k(A) \arrow[dl, "\epsilon_k"] \\
 & T(A) &
\end{tikzcd}$$

Figure 1: Direct limit.

Lemma 2.4. *The algebra $T(A)$ is a semisimple MV-algebra.*

Proof. Suppose that there exists a non-trivial infinitesimal element $\mathbf{x} \in T(A)$. It follows that $n\mathbf{x} \leq \mathbf{x}^*$ for any $n \in \mathbb{N}$, therefore $n\mathbf{x} \odot \mathbf{x} = \mathbf{0}$ for any $n \in \mathbb{N}$. This comes to the existence of naturals m,l,k such that $\mathbf{x}$ is the equivalence class of (x,m), $\mathbf{0}$ is the equivalence class of $(0,l)$ and $n\epsilon_{m,k}(x) \odot \epsilon_{m,k}(x) = \epsilon_{l,k}(0)$. Since $\mathbf{x} \neq \mathbf{0}$ and all maps are embeddings, $\epsilon_{m,k}(x) \neq 0$. This entails that $\epsilon_{m,k}(x)$ is a non-trivial infinitesimal in the semisimple MV-algebra $T^k(A)$, a contradiction. $\square$

Notation. For any $\mathbf{a} \in T^n(A)$ and any $\mathbf{b} \in T^m(A)$ in order to avoid confusion with the indexes of the disjoint union, we denote the product of functions used in Equation (1) as follows:

$$\gamma_{n,m} \colon T^n(A) \times T^m(A) \to T^{n+m}(A) \subseteq C(X^{n+m}),$$
$$\gamma_{n,m}(\mathbf{a},\mathbf{b})(x_1,\ldots,x_n,y_1,\ldots,y_m) = \mathbf{a}(x_1,\ldots,x_n)\cdot\mathbf{b}(y_1,\ldots,y_m).$$

The following lemma collects some technical properties of the maps $\epsilon_{n,m}$ and $\gamma_{n,m}$.

Lemma 2.5. *For any $n,m,k \in \mathbb{N}$, the following hold:*

(i) $\gamma_{n,m}(\mathbf{a},\mathbf{1}_m) = \epsilon_{n,n+m}(\mathbf{a})$, *with* $\mathbf{a} \in T^n(A)$ *and* $\mathbf{1}_m$ *top element in* $T^m(A)$, *that is, the unit function in* $C(X^m)$*;*

(ii) $\epsilon_{n+m} = \epsilon_{m+n}$ *and* $\epsilon_{n+(m+k)} = \epsilon_{(n+m)+k}$*;*

(iii) $\gamma_{n,m}(\mathbf{a},\mathbf{b}) = \gamma_{m,n}(\mathbf{b},\mathbf{a})$*, for any* $\mathbf{a} \in T^n(A)$ *and* $\mathbf{b} \in T^m(A)$*;*

(iv) $\gamma_{n,m+k}(\mathbf{a},\gamma_{m,k}(\mathbf{b},\mathbf{c})) = \gamma_{n+m,k}(\gamma_{n,m}(\mathbf{a},\mathbf{b}),\mathbf{c})$*, for any* $\mathbf{a} \in T^n(A)$*,* $\mathbf{b} \in T^m(A)$ *and* $\mathbf{c} \in T^k(A)$*;*

(v) If $n \leq m$*,* $\gamma_{m,k}(\epsilon_{n,m}(\mathbf{a}),\mathbf{b}) = \epsilon_{n+k,m+k}(\gamma_{n,k}(\mathbf{a},\mathbf{b}))$*.*

Proof. (i) For any $\mathbf{a} \in T^n(A)$, $\gamma_{n,m}(\mathbf{a},\mathbf{1}_m) = \mathbf{a}\cdot\mathbf{1}_m = \mathbf{a}\otimes\mathbf{1}_m = \epsilon_{n,n+m}(\mathbf{a})$.

(ii) By Proposition 2.1, $T^{n+m}(A) = T^{m+n}(A)$ and $\epsilon_{n+m,m+m}$ is the identity. Thus, for any $\mathbf{x} \in T^{n+m}(A)$ it follows from the commutativity of Figure 1 that $\epsilon_{n+m}(\mathbf{x}) = \epsilon_{m+n}(\epsilon_{n+m,m+n}(\mathbf{x})) = \epsilon_{m+n}(\mathbf{x})$. The second part of the claim follows with similar arguments.

(iii) We recall that any $T^k(A)$ is a subalgebra of $C(X^k)$. Moreover, $\gamma_{n,m}(\mathbf{a},\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} \in C(X^{n+m})$ and $\gamma_{m,n}(\mathbf{b},\mathbf{a}) = \mathbf{b}\cdot\mathbf{a} \in C(X^{m+n})$. Since $X^{n+m} \simeq X^{m+n}$, the conclusion follows by the commutativity of the product of functions.

(iv) By definition, $\gamma_{n,m+k}(\mathbf{a},\gamma_{m,k}(\mathbf{b},\mathbf{c})) = \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c}) \in C(X^{n+(m+k)})$ and $\gamma_{n+m,k}(\gamma_{n,m}(\mathbf{a},\mathbf{b}),\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c} \in C(X^{(n+m)+k})$. Since $X^{n+(m+k)} \simeq X^{(n+m)+k}$, the conclusion follows by the associativity of the product of functions.

(v) Using items (i), (iii) and (iv), we have:

$$\begin{aligned}
& \gamma_{m,k}(\epsilon_{n,m}(\mathbf{a}),\mathbf{b}) = \gamma_{(m-n)+n,k}(\gamma_{n,m-n}(\mathbf{a},\mathbf{1}_{m-n}),\mathbf{b}) = \\
= \quad & \gamma_{(m-n)+n,k}(\gamma_{m-n,n}(\mathbf{1}_{m-n},\mathbf{a}),\mathbf{b}) = \gamma_{m-n,n+k}(\mathbf{1}_{m-n},\gamma_{n,k}(\mathbf{a},\mathbf{b})) = \\
= \quad & \gamma_{n+k,m-n}(\gamma_{n,k}(\mathbf{a},\mathbf{b}),\mathbf{1}_{m-n}) = \epsilon_{n+k,m+k}(\gamma_{n,k}(\mathbf{a},\mathbf{b})).
\end{aligned}$$

□

Proposition 2.6. *For any semisimple MV-algebra* A*,* $T(A)$ *is a semisimple and unital PMV-algebra.*

Proof. We define the product as follows. For any $\mathbf{x},\mathbf{y} \in T(A)$ there exist $n, m \in \mathbb{N}$ such that $\mathbf{x} = \epsilon_n(\mathbf{a})$, with $\mathbf{a} \in T^n(A)$ and $\mathbf{y} = \epsilon_m(\mathbf{b})$, with $\mathbf{b} \in T^m(A)$. Then

$$\mathbf{x}\cdot\mathbf{y} = (\epsilon_{n+m}\circ\gamma_{n,m})(\mathbf{a},\mathbf{b}).$$

We first need to prove that the operation is well defined. Let $\mathbf{c} \in T^l(A)$ and $\mathbf{d} \in T^k(A)$ be elements such that $(\mathbf{a},n) \sim (\mathbf{c},l)$ and $(\mathbf{b},m) \sim (\mathbf{d},k)$. This means that we can assume, without loss of generality, $\mathbf{a} = \epsilon_{l,n}(\mathbf{c})$ and $\mathbf{b} = \epsilon_{k,m}(\mathbf{d})$. Then, using the commutativity of the diagram in Figure 1 and applying Lemma 2.5(v) twice, we get

$$\begin{aligned}
& \epsilon_{n+m}(\gamma_{n,m}(\mathbf{a},\mathbf{b})) = \epsilon_{n+m}(\gamma_{n,m}(\epsilon_{l,n}(\mathbf{c}),\epsilon_{k,m}(\mathbf{d}))) \\
= \quad & \epsilon_{n+m}(\epsilon_{m+l,m+n}(\gamma_{l,m}(\mathbf{c},\epsilon_{k,m}(\mathbf{d})))) = \epsilon_{m+l}(\gamma_{m,l}(\epsilon_{k,m}(\mathbf{d}),\mathbf{c})) \\
= \quad & \epsilon_{m+l}(\epsilon_{k+l,m+l}(\gamma_{l,k}(\mathbf{c},\mathbf{d}))) = \epsilon_{l+k}(\gamma_{l,k}(\mathbf{c},\mathbf{d})).
\end{aligned}$$

To prove that $T(A)$ is a PMV-algebra, let us prove that the function $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$ is bilinear, that the product is associative and that $\mathbf{1}$, the function identically equal to 1, is the unit.

To prove bilinearity[1], let $\mathbf{x_1}, \mathbf{x_2}, \mathbf{y}$ be elements in $T(A)$ such that

$$\mathbf{x_1} = \epsilon_n(\mathbf{a_1}) \text{ with } \mathbf{a_1} \in T^n(A), \qquad \mathbf{x_2} = \epsilon_m(\mathbf{a_2}) \text{ with } \mathbf{a_2} \in T^m(A)$$
$$\mathbf{x_1} + \mathbf{x_2} \text{ is defined}, \qquad \mathbf{y} = \epsilon_k(\mathbf{c}) \text{ with } \mathbf{c} \in T^k(A).$$

Without loss of generality, we assume that $n \leq m$. Therefore, since each of the ϵ-map is a homomorphism of MV-algebras,

$$\epsilon_n(\mathbf{a_1}) + \epsilon_m(\mathbf{a_2}) = \epsilon_m(\epsilon_{n,m}(\mathbf{a_1})) + \epsilon_m(\mathbf{a_2}) = \epsilon_m(\epsilon_{n,m}(\mathbf{a_1}) + \mathbf{a_2}).$$

Thus, by definition of the product on $T(A)$,

$$(\mathbf{x_1} + \mathbf{x_2}) \cdot \mathbf{y} = \epsilon_{m+k}(\gamma_{m,k}(\epsilon_{n,m}(\mathbf{a_1}) + \mathbf{a_2}, \mathbf{b}))$$

and since $\gamma_{m,k}$ is a bimorphism,

$$\gamma_{m,k}(\epsilon_{n,m}(\mathbf{a_1}) + \mathbf{a_2}, \mathbf{b}) = \gamma_{m,k}(\epsilon_{n,m}(\mathbf{a_1}), \mathbf{b}) + \gamma_{m,k}(\mathbf{a_2}, \mathbf{b}).$$

By Lemma 2.5(v) we have $\gamma_{m,k}(\epsilon_{n,m}(\mathbf{a_1}), \mathbf{b}) = \epsilon_{n+k,m+k}(\gamma_{n,k}(\mathbf{a_1}, \mathbf{b}))$ and

$$\begin{aligned}(\mathbf{x_1} + \mathbf{x_2}) \cdot \mathbf{y} =& \epsilon_{m+k}(\epsilon_{n+k,m+k}(\gamma_{n,k}(\mathbf{a_1}, \mathbf{b}))) + \epsilon_{m+k}(\gamma_{m,k}(\mathbf{a_2}, \mathbf{b})) = \\ =& \epsilon_{n+k}(\gamma_{n,k}(\mathbf{a_1}, \mathbf{b})) + \mathbf{x_2} \cdot \mathbf{y} = \mathbf{x_1} \cdot \mathbf{y} + \mathbf{x_2} \cdot \mathbf{y}.\end{aligned}$$

One can prove in the same way that $\mathbf{y} \cdot (\mathbf{x_1} + \mathbf{x_2}) = \mathbf{y} \cdot \mathbf{x_1} + \mathbf{y} \cdot \mathbf{x_2}$.

Associativity follows directly from Lemma 2.5 items (ii) and (iv).

Finally, for any $k \in \mathbb{N}$ we denote by $\mathbf{1}$ and $\mathbf{1}_k$ the top elements of $T(A)$ and $T^k(A)$ respectively. It follows that $\epsilon_k(\mathbf{1}_k) = \mathbf{1}$ for any $k \in \mathbb{N}$.

Let $\mathbf{x} \in T(A)$, such that $\mathbf{x} = \epsilon_n(\mathbf{a})$ with $\mathbf{a} \in T^n(A)$ and let m be a positive integer such that $\mathbf{1} = \epsilon_m(\mathbf{1}_m)$. We have

$$\mathbf{x} \cdot \mathbf{1} = \epsilon_{n+m}(\gamma_{n,m}(\mathbf{a}, \mathbf{1}_m)) = \epsilon_{n+m}(\epsilon_{n,n+m}(\mathbf{a})) = \epsilon_n(\mathbf{a}) = \mathbf{x}.$$

The proof of the equality $\mathbf{1} \cdot \mathbf{x} = \mathbf{x}$ follows from Lemma 2.5 items (i) and (iii). Then $T(A)$ is a unital PMV-algebra whose MV-algebra reduct is semisimple, and this entails that $T(A)$ is a unital and semisimple PMV-algebra. ☐

Theorem 2.7. *For any semisimple MV-algebra A, for any semisimple and unital PMV-algebra P and for any homomorphism of MV-algebras $f : A \to \mathcal{U}_{(\cdot)}(P)$ there is a homomorphism of PMV-algebras $\widetilde{f} : T(A) \to P$ such that $\widetilde{f} \circ \epsilon_{1,A} = f$.*

[1] We recall that in order to prove linearity of a map $\beta : A \to B$, one needs to show that $\beta(a_1 \oplus a_2) = \beta(a_1) \oplus \beta(a_2)$ for any a_1, a_2 such that $a_1 \leq a_2^*$. To give a more compact version of the statement, it is possible to define a partial sum $+$ by "$a_1 + a_2$ is defined iff $a_1 \leq a_2^*$ and in this case $a_1 + a_2 = a_1 \oplus a_2$". Thus, linearity of β equals to $\beta(a_1 + a_2) = \beta(a_1) + \beta(a_2)$.

Proof. Using the universal property of the tensor product, let us define the following family of homomorphisms $\{\widetilde{\lambda_n}\}_{n\in\mathbb{N}}$.

(i) For $n = 1$, $\lambda_1 = \widetilde{\lambda}_1 = f$.

(ii) For $n = 2$, we define $\lambda_2 : A \times A \to P$ to be the function such that $\lambda_2(a_1, a_2) = f(a_1) \cdot f(a_2)$. Since P is a unital PMV-algebra, λ_2 is a bimorphism and $\lambda_2(1_A, 1_A) = f(1_A) \cdot f(1_A) = 1_P \cdot 1_P = 1_P$. Whence, there exists a homomorphism of MV-algebras $\widetilde{\lambda_2} : A \otimes A \to A$ such that $\widetilde{\lambda}_2(a_1 \otimes a_2) = f(a_1) \cdot f(a_2)$.

(iii) for any $n \in \mathbb{N}$, $\lambda_n : T^{n-1}(A) \times A \to P$, $\lambda_n(\mathbf{x}, a_n) = \widetilde{\lambda}_{n-1}(\mathbf{x}) \cdot f(a_n)$. Thus, $\widetilde{\lambda}_n : T^n(A) \to P$ is the homomorphism such that $\widetilde{\lambda}_n(a_1 \otimes \ldots \otimes a_n) = f(a_1) \cdot \ldots \cdot f(a_n)$.

It is easily seen that $\widetilde{\lambda}_m \circ \epsilon_{n,m} = \widetilde{\lambda}_n$ for any $n \leq m$, since the two homomorphisms coincide on generators. It follows from [4, Chapter 3, §7.6, Proposition 6] that $(T(A), \epsilon_n)$ satisfies the universal property of Figure 2.

$$\begin{tikzcd}
T^n(A) \arrow[rr, "\epsilon_{n,m}"] \arrow[dr, "\epsilon_n"] \arrow[ddr, "\widetilde{\lambda}_n"'] & & T^m(A) \arrow[dl, "\epsilon_m"'] \arrow[ddl, "\widetilde{\lambda}_m"] \\
& T(A) \arrow[d, dashed] & \\
& P &
\end{tikzcd}$$

Figure 2: Universal property.

Whence, there exists a homomorphism of MV-algebras $\widetilde{f} : T(A) \to P$ such that $\widetilde{f} \circ \epsilon_n = \widetilde{\lambda}_n$, and then $\widetilde{f} \circ \epsilon_1 = \widetilde{\lambda_1} = f$.
Finally, the fact that $\widetilde{f}$ is a homomorphism of PMV-algebras is a direct consequence of [3, Theorem 1.1], taking into account the fact that unital and semisimple PMV-algebras correspond to unital and Archimedean f-rings, and that such rings are semiprime (that is, without non-trivial nilpotent elements). $\square$

Corollary 2.8. *Let A and B be semisimple MV-algebras and $h : A \to B$ be a homomorphism of MV-algebras. There exists a unique homomorphism of PMV-algebras $h^\sharp : T(A) \to T(B)$ such that $h^\sharp \circ \epsilon_{1,A} = \epsilon_{1,B} \circ h$.*

Proof. It follows from Theorem 2.7, choosing $f = \epsilon_{1,B} \circ h$. $\square$

Applying the construction of the Tensor PMV-algebra to a Riesz MV-algebra, we obtain the following result.

Theorem 2.9. *If A is a semisimple Riesz MV-algebra, then $T(A)$ is a unital and semisimple fMV-algebra.*

Proof. By [16, Proposition 3.1], any $T^n(A)$ is a Riesz MV-algebra. If $\mathbf{x} \in T(A)$, there exist $n \in \mathbb{N}$ and $\mathbf{a} \in T^n(A)$ such that $\mathbf{x} = \epsilon_n(\mathbf{a})$. We define the scalar product on the direct limit as

$$\alpha\mathbf{x} = \epsilon_n(\alpha\mathbf{a}), \text{ for any } \alpha \in [0,1].$$

Let us first prove that the operation is well defined. If $\mathbf{x} = \epsilon_n(\mathbf{a}) = \epsilon_m(\mathbf{b})$, and $n \leq m$, by Remark 2.3 and the injectivity of ϵ_m, it follows that $\mathbf{b} = \epsilon_{n,m}(\mathbf{a})$. Since each $\epsilon_{n,m}$ is a homomorphism of Riesz MV-algebras (see [23, Proposition 11.53], where the claim is proved for Archimedean Riesz spaces), we get

$$\epsilon_m(\alpha\mathbf{b}) = \epsilon_m(\alpha\epsilon_{n,m}(\mathbf{a})) = \epsilon_m(\epsilon_{n,m}(\alpha\mathbf{a})) = \epsilon_n(\alpha\mathbf{a}).$$

Take now $\mathbf{x} = \epsilon_n(\mathbf{a})$, $\mathbf{y} = \epsilon_m(\mathbf{b})$ and, without loss of generality, assume $n \leq m$. If the partial sum is defined, we get

$$\mathbf{x} + \mathbf{y} = \epsilon_n(\mathbf{a}) + \epsilon_m(\mathbf{b}) = \epsilon_m(\epsilon_{n,m}(\mathbf{a})) + \epsilon_m(\mathbf{b}) = \epsilon_m(\epsilon_{n,m}(\mathbf{a}) + \mathbf{b}).$$

As any $T^n(A)$ is a Riesz MV-algebra, and since any $\epsilon_{n,m}$ is a homomorphism of Riesz MV-algebras, we infer:

(i) $\alpha(\mathbf{x} + \mathbf{y}) = \epsilon_m(\alpha(\epsilon_{n,m}(\mathbf{a}) + \mathbf{b})) = \epsilon_m(\epsilon_{n,m}(\alpha\mathbf{a})) + \epsilon_m(\alpha\mathbf{b}) =$
$= \epsilon_n(\alpha\mathbf{a}) + \epsilon_m(\alpha\mathbf{b}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.

(ii) $(\alpha + \beta)\mathbf{x} = \epsilon_n((\alpha + \beta)\mathbf{a}) = \epsilon_n(\alpha\mathbf{a} + \beta\mathbf{a}) = \epsilon_n(\alpha\mathbf{a}) + \epsilon_n(\beta\mathbf{a}) = \alpha\mathbf{x} + \beta\mathbf{x}$.

(iii) $(\alpha \cdot \beta)\mathbf{x} = \epsilon_n((\alpha \cdot \beta)\mathbf{a}) = \epsilon_n(\alpha(\beta\mathbf{a})) = \alpha\mathbf{y}$, with $\mathbf{y} = \epsilon_n(\beta\mathbf{a}) = \beta\mathbf{x}$.

(iv) $1\mathbf{x} = \epsilon_n(1\mathbf{a}) = \epsilon_n(\mathbf{a}) = \mathbf{x}$.

Hence, $T(A)$ is a unital PMV-algebra and a Riesz MV-algebra. Using the fact that each $T^n(A)$ is an algebra of functions, we deduce the associativity law required in the definition of an fMV-algebra:

$$\begin{aligned} & \alpha(\mathbf{x} \cdot \mathbf{y}) = \epsilon_{n+m}(\alpha\gamma_{n,n}(\mathbf{a}, \mathbf{b})) = \epsilon_{n+m}(\alpha(\mathbf{a} \cdot \mathbf{b})) = \epsilon_{n+m}((\alpha\mathbf{a}) \cdot \mathbf{b})) = \\ = \; & \epsilon_{n+m}(\gamma_{n,m}(\alpha\mathbf{a}, \mathbf{b})) = (\alpha\mathbf{x}) \cdot \mathbf{y}. \end{aligned}$$

In the same way we prove that $\alpha(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot (\alpha\mathbf{y})$, settling the claim. □

Theorem 2.10. *If A is a unital and semisimple PMV-algebra, then $A \simeq T(A)$.*

Proof. We first remark that, A being a PMV-algebra, for any $n \in \mathbb{N}$,

$$T^n(A) = \langle f_1 \cdot \ldots \cdot f_n \mid f_i \in A \subseteq C(X) \rangle \subseteq A = T^1(A).$$

Each $T^n(A)$ is therefore an MV-subalgebra of A but in general it is not a PMV-subalgebra. Moreover, since $T^n(A) \subseteq A$, for any $n \in \mathbb{N}$ the embedding $\epsilon_{1,n}$ is an isomorphism of MV-algebras. Let us prove that the embedding $\epsilon_1 : A \to T(A)$ gives the desired isomorphism. Let $\mathbf{y}$ be an element of $T(A)$, with $\mathbf{y} = \epsilon_n(\mathbf{a})$, for some $n \in \mathbb{N}$ and $\mathbf{a} \in T^n(A)$. Then $\mathbf{a} = \epsilon_{1,n}(\mathbf{b})$, for $\mathbf{b} \in A$ and $\mathbf{y} = \epsilon_n(\mathbf{a}) = \epsilon_n(\epsilon_{1,n}(\mathbf{b})) = \epsilon_1(\mathbf{b})$, that is, ϵ_1 is surjective. Finally, from [3, Theorem 1.1] we deduce that ϵ_1 is an isomorphism of PMV-algebras. □

Remark 2.11. A first attempt at making this construction without the requirement of semisimplicity can be found in [18]. The main proof was based on [12, Theorem 4.11], which turned out to contain a mistake, see [16, Remark 3.1].

2.1. Two applications

Applying the results of this section, we prove the amalgamation property for semisimple algebras and we characterize free objects in all three cases.

Proposition 2.12. $\mathbf{uPMV_{ss}}$, $\mathbf{ufMV_{ss}}$ *and* $\mathbf{RMV_{ss}}$ *have the amalgamation property.*

Proof. We give the complete proof for $\mathbf{uPMV_{ss}}$. Let A, B, Z be unital and semisimple PMV-algebras such that Z embeds in both A and B, with embeddings z_A and z_B.

Consider the MV-reducts of A, B and Z. By [22, Theorem 2.20], there exist an MV-algebra C and embeddings f_A, f_B such that $f_A : A \hookrightarrow C$, $f_B : B \hookrightarrow C$. Since we deal with semisimple algebras, it is easy to prove that $\pi \circ f_A$ and $\pi \circ f_B$ are embeddings of A and B in $D = C/Rad(C)$, where $\pi : C \to C/Rad(C)$ is the canonical epimorphism.

By Remark 2.3, the algebra $D = T^1(D)$ embeds in $T(D)$ via $\epsilon_{1,D}$. We get two embeddings $\overline{f_A} : A \hookrightarrow T(D)$ and $\overline{f_B} : B \hookrightarrow T(D)$, where $\overline{f_A} = \epsilon_{1,D} \circ \pi \circ f_A$ and $\overline{f_B} = \epsilon_{1,D} \circ \pi \circ f_B$.

$$\begin{tikzcd}
& A \arrow[dr, hook, "\pi \circ f_A"] & & \\
Z \arrow[ur, hook, "z_A"] \arrow[dr, hook, "z_B"'] & & D = C \big/ Rad(C) \arrow[r, hook, "\epsilon_{1,D}"] & T(D) \\
& B \arrow[ur, hook, "\pi \circ f_B"'] & &
\end{tikzcd}$$

Figure 3: Amalgamation property.

Since $f_A \circ z_A = f_B \circ z_B$ the diagram commutes, and by [3, Theorem 1.1] we can deduce that $\overline{f_A}$ and $\overline{f_B}$ are homomorphisms of PMV-algebras, settling the claim.

The proofs for $\mathbf{ufMV_{ss}}$ and $\mathbf{RMV_{ss}}$ are similar: we shall further embed $T(D)$ into $[0,1]\otimes T(D)$ in the case of fMV-algebras, and D directly into $[0,1]\otimes D$ in the case of Riesz MV-algebras. Then, using [15, Proposition 3.2] and [23, Proposition 11.53] respectively, we settle the claim. □

If we restrict our attention to the varieties generated by $[0,1]$, the free MV(Riesz MV, PMV, fMV)-algebra k-generated exists and it is the subalgebra of $[0,1]^{[0,1]^k}$ generated by the projection maps. The standard MV-algebra $[0,1]$ plays a different role in each instance, and for clarity we shall denote the standard models in the corresponding varieties by

$$[0,1]_{MV} = ([0,1], \oplus, ^*, 0) \qquad [0,1]_{RMV} = ([0,1], \oplus, ^*, \{\alpha\}_{\alpha\in[0,1]}, 0)$$
$$[0,1]_{PMV} = ([0,1], \oplus, ^*, \cdot, 0) \qquad [0,1]_{fMV} = ([0,1], \oplus, ^*, \cdot, \{\alpha\}_{\alpha\in[0,1]}, 0).$$

Let $Free_{MV}(n)$ and $Free_{RMV}(n)$ be the free MV-algebra and, respectively, the free Riesz MV-algebra over n generators. Let $Free_{fMV}(n)$ be the free fMV-algebra over n generators in HSP($[0,1]_{fMV}$), and let $Free_{PMV}(n)$ be the free PMV-algebra over n generators in HSP($[0,1]_{PMV}$). We recall that the variety of Riesz MV-algebras is generated by the standard model $[0,1]$, while in the case of PMV-algebras and fMV–algebras, the algebra $[0,1]$ generates proper subvarieties [19, 15].

Proposition 2.13. *For $n \in \mathbb{N}$, the following hold:*

(i) $Free_{RMV}(n) \simeq [0,1]_{RMV} \otimes Free_{MV}(n)$,

(ii) $Free_{PMV}(n) \simeq T(Free_{MV}(n))$,

(iii) $Free_{fMV}(n) \simeq [0,1]_{RMV} \otimes T(Free_{MV}(n)) \simeq T([0,1]_{RMV} \otimes Free_{MV}(n))$.

Proof. (i) It is [16, Proposition 5.1(i)].

(ii) Let P be a unital and semisimple PMV-algebra and let $f : X \to P$ be a function, with $|X| = n$. There is a unique homomorphism of MV-algebras $f^{\#} : Free_{MV}(n) \to \mathcal{U}_{(\cdot)}(P)$ that extends f. Free algebras being semisimple, Theorem 2.7 ensures that there exists a homomorphism of PMV-algebras $\widetilde{f} : T(Free_{MV}(n)) \to P$ such that $\widetilde{f} \circ \epsilon_{1,Free_{MV}(n)} = f^{\#}$. The uniqueness of $\widetilde{f}$ is a consequence of the uniqueness of $f^{\#}$. Since $\epsilon_{1,Free_{MV}(n)}$ is an embedding we have $X \simeq \epsilon_{1,Free_{MV}(n)}(X)$ and $T(Free_{MV}(n))$ is the free object in $\mathbf{uPMV_{ss}}$. Since $Free_{PMV}(n)$ is also an object in $\mathbf{uPMV_{ss}}$, we deduce that $T(Free_{MV}(n)) \simeq Free_{PMV}(n)$.

(iii) The first isomorphism follows from (ii) and [16, Proposition 5.1(ii)]. The second isomorphism can be proved using (i) and a similar argument to the one used in (ii). $\square$

3. A category-theoretical perspective

In this section we connect the categories of semisimple MV-algebras, unital and semisimple PMV-algebras and unital and semisimple fMV-algebras by adjuctions. Using the notations of Table 2, we start this section by defining the following maps.

(i) The map $\mathbf{T} : \mathbf{MV_{ss}} \to \mathbf{uPMV_{ss}}$ is defined as follows.

- (a) For any $A \in \mathbf{MV_{ss}}$, $\mathbf{T}(A)$ is the tensor PMV-algebra $T(A)$. By Proposition 2.6 it is a unital and semisimple PMV-algebra.
- (b) For any homomorphism of MV-algebras $h : A \to B$, $\mathbf{T}(h)$ is the homomorphism of PMV-algebras $h^{\sharp}$ defined in Corollary 2.8.

(ii) The map $\mathcal{F}_{\mathbf{T}} : \mathbf{RMV_{ss}} \to \mathbf{ufMV_{ss}}$ is defined as follows.

- (a) For any $R \in \mathbf{RMV_{ss}}$, $\mathcal{F}_{\mathbf{T}}(R)$ is the tensor PMV-algebra $T(R)$. By Theorem 2.9 it is a unital and semisimple fMV-algebra.

(b) For any homomorphism of Riesz MV-algebras $h : R_1 \to R_2$, $\mathcal{F}(h)$ is the homomorphism $h^\sharp$ defined in Proposition 2.8. It is a homomorphism of fMV-algebras by [10, Corollary 3.11].

(iii) From $\mathbf{uPMV_{ss}}$ to $\mathbf{MV_{ss}}$ and from $\mathbf{ufMV_{ss}}$ to $\mathbf{RMV_{ss}}$ we consider the usual forgetful functors, both denoted by $\mathcal{U}_{(\cdot)}$.

Lemma 3.1. *The maps* $\mathbf{T}$ *and* $\mathcal{F}_{\mathbf{T}}$ *are functors.*

Proof. We give the proof in the case of $\mathbf{T}$, the other one being similar.

Denoted by $\mathbf{I}_A$ and $\mathbf{I}_{\mathbf{T}(A)}$ the identity maps on A and $\mathbf{T}(A)$ respectively, it is easy to check that $\mathbf{I}_{\mathbf{T}(A)} \circ \epsilon_{1,A} = \epsilon_{1,A} \circ \mathbf{I}_A$, therefore $\mathbf{I}_A^\sharp = \mathbf{I}_{\mathbf{T}(A)}$.

Let $h : A \to B$ and $g : B \to C$ be homomorphisms of MV-algebras. We have

$$\begin{aligned}(g^\sharp \circ h^\sharp) \circ \epsilon_{1,A} =& g^\sharp \circ (h^\sharp \circ \epsilon_{1,A}) = \\ =& g^\sharp \circ (\epsilon_{1,B} \circ h) = (g^\sharp \circ \epsilon_{1,B}) \circ h = \epsilon_{1,C} \circ (g \circ h),\end{aligned}$$

then $(g^\sharp \circ h^\sharp) = (g \circ h)^\sharp$ and $\mathbf{T}$ is a functor. □

Lemma 3.2. *The families of maps* $\{\epsilon_{1,A}\}_{A\in\mathbf{MV_{ss}}}$ *and* $\{\epsilon_{1,R}\}_{R\in\mathbf{RMV_{ss}}}$ *are a natural transformation between the identity functor on* $\mathbf{MV_{ss}}$ *and the composite functor* $\mathcal{U}_{(\cdot)} \circ \mathbf{T}$*, and the identity functor on* $\mathbf{RMV_{ss}}$ *and the composite functor* $\mathcal{U}_{(\cdot)} \circ \mathcal{F}_{\mathbf{T}}$*, respectively.*

Proof. We give details for the first case. Let $h : A \to B$ be a homomorphism of MV-algebras. We need to prove that $(\mathcal{U}_{(\cdot)} \circ \mathbf{T})(h) \circ \epsilon_{1,A} = \epsilon_{1,B} \circ h$. Since $(\mathcal{U}_{(\cdot)} \circ \mathbf{T})(h) = h^\sharp$, the result follows from Corollary 2.8. □

Theorem 3.3. *The pairs of functors* $(\mathbf{T},\mathcal{U}_{(\cdot)})$ *and* $(\mathcal{F}_{\mathbf{T}},\mathcal{U}_{(\cdot)})$ *establish two adjuctions.*

Proof. In the case of the first pair of functors, it follows from Theorem 2.7 and Lemma 3.2 that for any unital and semisimple PMV-algebra P and any homomorphism of MV-algebras $f : A \to \mathcal{U}_{(\cdot)}(P)$, with $A \in \mathbf{MV_{ss}}$, there exists a homomorphism of PMV-algebras $f^\sharp : \mathbf{T}(A) \to P$ such that $\mathcal{U}_{(\cdot)}(f^\sharp) \circ \iota_A = f$. This implies that $\mathbf{T}$ is a left adjoint functor for $\mathcal{U}_{(\cdot)}$, and the claim is settled. □

Using Theorem 3.3 and the results from [16], we obtain two different paths from semisimple MV-algebras to unital and semisimple fMV-algebras, as displayed in Figure 4.

Proposition 3.4. *The composite functors* $\mathcal{F}_{\mathbf{T}} \circ \mathcal{T}_{\otimes}$ *and* $\mathcal{F}_{\otimes} \circ \mathbf{T}$ *are naturally isomorphic.*

Proof. It follows from well-known properties of adjoint functors, since the composite functors $\mathcal{F}_{\mathbf{T}} \circ \mathcal{T}_{\otimes}$ and $\mathcal{F}_{\otimes} \circ \mathbf{T}$ are both adjoints of the forgetful functor from $\mathbf{ufMV_{ss}}$ to $\mathbf{MV_{ss}}$. □

$$
\begin{tikzcd}
\mathbf{MV_{ss}} \arrow[r, "\mathcal{T}_{\otimes}"] \arrow[d, "\mathbf{T}"'] & \mathbf{RMV_{ss}} \arrow[d, "\mathcal{F}_{\mathbf{T}}"] \\
\mathbf{uPMV_{ss}} \arrow[r, "\mathcal{F}_{\otimes}"] & \mathbf{ufMV_{ss}}
\end{tikzcd}
$$

Figure 4: From MV-algebras to fMV-algebras.

3.1. From MV-algebras to ℓu-groups

As mentioned in Section 1, semisimple MV-algebras are categorical equivalent to Archimedean ℓu-groups, unital and semisimple PMV-algebras are categorical equivalent to unital and Archimedean ℓu-rings endowed with a strong unit such that $u \cdot u \leq u$, unital and semisimple fMV-algebras are categorical equivalent to unital and Archimedean fu-algebras endowed with a strong unit such that $u \cdot u \leq u$. Moreover, the tensor product of Archimedean ℓu-groups is defined and investigated in [11, 5]. If we denote this tensor product by $\otimes_a$, the following property is proved in [16]: if (G_A, u_A), (G_B, u_B) are Archimedean ℓu-groups and A, B are semisimple MV-algebras such that $A \simeq \Gamma(G_A, u_A)$ and $B \simeq \Gamma(G_B, u_B)$, then

$$A \otimes B \simeq \Gamma(G_A \otimes_a G_B, u_A \otimes_a u_B). \tag{2}$$

Building on Equation (2), all results of Sections 2 and 3 can be transferred to ℓu-groups, ℓu-rings and fu-algebras. If Ξ is the inverse functor of Γ, $\Xi_{(\cdot)}$ is the inverse functor of $\Gamma_{(\cdot)}$ and $A = \Gamma(G, u)$, then $(R, v) = \Xi_{(\cdot)}(T(A))$ will be the unital and Archimedean *tensor fu-ring of* (G, u). We will subsequently denote it by $T(G, u)$.

Applying the inverses of the functors Γ and $\Gamma_{(\cdot)}$, the adjunction given by $(\mathbf{T}, \mathcal{U}_{(\cdot)})$ can be lifted to an adjunction $(\mathbf{T_a}, \mathcal{U}_{(\cdot\ell)})$ between $\mathbf{auG_a}$ and $\mathbf{uR_a}$. In the same way, the adjunction $(\mathcal{F}_{\mathbf{T}}, \mathcal{U}_{(\cdot)})$ is lifted to an adjunction $(\mathcal{F}_{\mathbf{T_a}}, \mathcal{U}_{(\cdot\ell)})$ between $\mathbf{uRS_a}$ and $\mathbf{fuAlg_a}$. Expanding the diagram of Figure 4, in Figure 5 we present the complete diagram that displays vertically the four adjuctions obtained in the current work and horizontally the four adjunctions obtained in [16]. One can easily verify that $T(G, u)$ can be equivalently obtained as direct limit of the direct system $(T^n(G, u), \widetilde{\epsilon_{n,m}})$, where $T^n(G, u) = G \otimes_a \ldots \otimes_a G$, n times, and $\widetilde{\epsilon_{n,m}}$ are the analogous of the embeddings given in Definition 2.2. We also point out that the tensor product of f-algebras is defined in [1, 6]. Via categorical equivalence, there is overlap between [1, 6] and [16], where it was already proved that, in the case of f-algebras with a strong unit, the Archimedean tensor product of ℓu-groups preserves the algebraic structure of the factors.

We conclude remarking that in the Archimedean setting, building on the deep connection between the theory of MV-algebras and the theory of lattice-

$$
\begin{tikzcd}
\mathbf{auG_a} \arrow[rrr, bend left, "\mathcal{T}_{\otimes a}"] \arrow[dr, leftrightarrow, "\Gamma"] \arrow[ddd, bend right, "\mathbf{T_a}"'] & & & \mathbf{uRS_a} \arrow[lll, "\mathcal{U}_{(\ell\mathbb{R})}"] \arrow[dl, leftrightarrow, "\Gamma_{\mathbb{R}}"'] \arrow[ddd, bend left, "\mathcal{F}_{\mathbf{T_a}}"] \\
& \mathbf{MV_{ss}} \arrow[r, bend left, "\mathcal{T}_{\otimes}"] \arrow[d, bend right, "\mathbf{T}"'] & \mathbf{RMV_{ss}} \arrow[l, "\mathcal{U}_{\mathbb{R}}"] \arrow[d, bend left, "\mathcal{F}_{\mathbf{T}}"] & \\
& \mathbf{uPMV_{ss}} \arrow[u, "\mathcal{U}_{(\cdot)}"'] \arrow[r, bend right, "\mathcal{F}_{\otimes}"'] & \mathbf{ufMV_{ss}} \arrow[l, "\mathcal{U}_{\mathbb{R}}"'] \arrow[u, "\mathcal{U}_{(\cdot)}"] & \\
\mathbf{uR_a} \arrow[uuu, "\mathcal{U}_{(\cdot\ell)}"'] \arrow[ur, leftrightarrow, "\Gamma_{(\cdot)}"] \arrow[rrr, bend right, "\mathcal{F}_{\otimes a}"'] & & & \mathbf{ufAlg_a} \arrow[lll, "\mathcal{U}_{(\ell\mathbb{R})}"'] \arrow[ul, leftrightarrow, "\Gamma_f"'] \arrow[uuu, "\mathcal{U}_{(\cdot\ell)}"]
\end{tikzcd}
$$

Figure 5: The complete diagram.

ordered groups with strong unit (that is extended to the whole algebraic hierarchy from groups to algebras), the tensor product proves to be a valuable construction that connects the prominent lattice-ordered algebras related to Łukasiewicz logic with product.

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