de Finetti's coherence and exchangeability in infinitary logic

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Abstract

We continue the investigation towards a logic-based approach to statistics within the infinitary conservative extension of Łukasiewicz logic $\mathcal{IRL}$ and prove versions of de Finetti's theorems on coherence and exchangeability. In particular we will prove a coherence criterion for a subclass of the variety of σ-complete Riesz MV-algebras in the conditional and unconditional case, and discuss de Finetti's exchangeability in a special case.

Keywords: de Finetti, coherence criterion, exchangeability, random variables, Baire functions, infinitary logic

1. Introduction

Probability theory and fuzzy logic are both theories used when one aims at performing some sort of inference in uncertain or imprecise situations. These two theories have been combined in many different ways and for the purpose of this paper we only recall [4], where the authors define the algebraic counterpart of a random variable within a conservative expansion of Łukasiewicz logic. The probabilistic setting used there is the one of subjective probability, as introduced by Bruno de Finetti.

Starting from the Thirties, Bruno de Finetti layed the ground for the development of subjective probability (also called Bayesian) in form of a betting game: In his setting, the probability of an event is the amount that a rational agent is willing to bet on it. de Finetti also proved that this point of view on probability is consistent with the widely used axiomatic approach. Indeed, it is possible to define a notion of *coherence* for the choices of the bets on the events and to prove that a book is coherent if, and only if, it can be extended to a probability measure on the Boolean algebra generated by the events. Coherence is therefore a bridge between subjective and objective, axiomatic, probability.

Another point of view on probability theory is the so-called frequentist approach. In this case, the probability of an event is defined by the frequency of

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that event based on previous observations. In this setting, it is common to define a statistical model that fits the observed data, and to derive the properties of the hidden probability distribution in this way. Perhaps surprisingly, subjective probability is again related to frequentist probability via de Finetti's work, and in particular via the notion of *exchangeability* that, loosely speaking, shows how statistical models appear in a Bayesian framework, and how probabilities can come from statistics, see [11, Chapter 5]. Formally, the exchangeability of a sequence of random variables means that the joint distribution of any of its finite subsequences is independent of any permutation of the order, see Section 2.3.

In this work, we want to continue the research project started in [4, 16], whose long-term goal is to provide a *metamathematics of statistics*, meant as a logic-based approach to the subject. To move towards this objective, it is necessary to enlighten the most suitable logical system in which one can work, and for us this is a system based on Łukasiewicz logic, since it has already be proven to be adequate to reason about subjective probability. Indeed, as one can appreciate in [8, 21], Łukasiewicz logic (via its algebraic semantics, the variety of MV-algebras) allows to codify probability measures in algebraic terms, using the notion of a *state*, see Section 2.3 for further details. Furthermore, such a logical system models the fundamental operations that one need to discuss probability: a sum and a complement to 1.

In particular, among all the possible expansions of Łukasiewicz logic, we shall work in the system defined in [6] which is an infinitary logic that allows to further model a scalar multiplication by real numbers as well as countable suprema. These are indeed natural operations to consider when it comes to reason about probability, for example, by taking convex combinations or discussing the probability of an increasing sequence of events. More formally, we will work in the logical system $\mathcal{IRL}$, that contains one operation of countable arity but it turned out to be extremely well-behaved. Indeed, it is a conservative extension of Łukasiewicz logic and its algebraic semantics is given by a special subclass of MV-algebras, namely the class of σ-complete Riesz MV-algebras, that were proved to be an infinitary variety in [6]. Moreover, $\mathcal{IRL}$ is standard complete: a formula of $\mathcal{IRL}$ is true in any σ-complete Riesz MV-algebra if, and only if, it is true in $[0, 1]$.

To continue our investigation towards a metamathematics of statistics within the system $\mathcal{IRL}$, de Finetti's theorems on coherence and exchangeability are crucial. Thus, in this note we will prove a coherence criterion for a subclass of the variety of σ-complete Riesz MV-algebras and discuss exchangeability in a special case. In particular, in Section 3 we discuss the structure of the spaces of $[0, 1]$-valued σ-homomorphisms and states of a σ-complete Riesz MV-algebra. In Section 4 we prove a suitable version of the coherence criterion, with a proof strategy that is different from the one usually given for MV-algebras, which couldn't been applied directly in this case. The coherence criterion is also discussed in the case of conditional events. Finally, in Section 5, we discuss exchangeability of a sequence of *boolean observables* via Hausdorff's moment problem. Here, by "observable" is meant the generalization of the notion of random variable given in [4], see Section 2.3 for further details.

2. Preliminaries

2.1. σ-complete Riesz MV-algebras

In this paper we will deal with *σ-complete Riesz MV-algebras*. These are algebras that model an infinitary and conservative extension of Łukasiewicz logic, called *Infinitary Riesz Logic* and denoted by $\mathcal{IRL}$, and can be thought of as unit intervals of Dedekind σ-complete vector lattices with strong unit. The logic $\mathcal{IRL}$ has been defined in [6] and σ-complete Riesz MV-algebras have been investigated as an infinitary class of algebras in [6, 5].

More precisely, a σ-complete Riesz MV-algebra is an algebra

$$\left(A, \oplus, \neg, 0, 1, \{\alpha\}_{\alpha\in[0,1]}, \bigvee\right),$$

where $\oplus$ is a binary operation, $\neg$ is an involution, 0 and 1 are respectively a bottom and a top element, the unary operations $\{\alpha\}_{\alpha\in[0,1]}$ model a scalar multiplication, and the infinitary operation $\bigvee$ models a countable disjunction. An additional operation can be defined as follows: $x \odot y := \neg(\neg a \oplus \neg y)$. This definition is willingly imprecise, since we shall soon restrict our attention on a special subclass of these algebras.

The standard example of such an algebra is the real interval $[0, 1]$, where $x \oplus y = \min(x + y, 1)$, $\neg x = 1 - x$, αx is the product of real numbers, and $\bigvee_n x_n$ is the supremum in $[0, 1]$. This example is standard in a very precise sense: σ-complete Riesz MV-algebras form an infinitary variety (see [24]) and $[0, 1]$ is a generator for it. This variety will be denoted $\mathbf{RMV}_\sigma$.

For any $A \in \mathbf{RMV}_\sigma$, we call *σ-ideals* the ideals of A that are closed under countable suprema, while by *MV-maximal σ-ideals* we mean those σ-ideals of A that are also maximal ideals for the Riesz MV-reduct of A. The set of all MV-maximal σ-ideals will be denoted by $\mathcal{M}_\sigma(A)$, or $\mathcal{M}_\sigma$ when A is clear from the context.

From a different point of view, if (V, u) is a Dedekind σ-complete Riesz space with a distinguished strong order unit u, the interval $[0, u]_V = \{x \in V \mid 0 \leq x \leq u\}$ is a σ-complete Riesz MV-algebra when endowed with the following operations: $x \oplus y = (x +_V y) \wedge u$, $\neg x = u -_V x$, αx and $\bigvee_n x_n$ the same as in V. The map that takes (V, u) and sends it into $[0, u]_V$ is actually a functor, the so-called Mundici's functor denoted by Γ, that induces a categorical equivalence.

2.2. Free objects and σ-semisimple algebras

Given a topological space (X, τ), $C(X)$ will denote the set of $[0, 1]$-valued continuous functions defined over X. A *zeroset* $Z \subseteq X$ is a set for which there exists $f \in C(X)$ such that $Z = \{x \in X \mid f(x) = 0\}$. A *cozero* set is a complement of a zeroset, that is a set A for which there exists a continuous function f such that $A = \{x \in X \mid f(x) \neq 0\}$.

A *Baire set* is a subset of X belonging to the σ-algebra generated by the zerosets of functions in $C(X)$, while a *Borel set* is a subset of X belonging to the σ-algebra generated by the closed sets (equivalently, by cozero subsets and open subsets respectively). We shall denote the σ-algebras of Baire and Borel subsets of X respectively by $\mathcal{BA}(X)$ and $\mathcal{BO}(X)$. Whence, a Baire function is a function $f\colon X \to [0,1]$ measurable with respect to the spaces $(X, \mathcal{BA}(X))$, $([0,1], \mathcal{BA}([0,1]))$. Borel functions are analogously defined. Note that $\mathcal{BA}(X) \subseteq \mathcal{BO}(X)$, while the converse inclusion holds for a metrizable space, see [5, Remark 2.1]. In particular, $\mathcal{BO}([0,1]^\kappa) = \mathcal{BA}([0,1]^\kappa)$ for $\kappa \leq \omega$, and $[0,1]^\kappa$ is endowed with the standard Euclidean topology.

It is known that the sets of $[0,1]$-valued Baire and Borel functions defined over some hypercube $[0,1]^\kappa$ are σ-complete Riesz MV-algebras, and they shall be denoted respectively by Baire$([0,1]^\kappa)$ and Borel$([0,1]^\kappa)$. We also recall that in both algebras countable suprema are taken pointwise.

In [5], it was proved that the free κ-generated algebra in $\mathbf{RMV}_\sigma$ coincides with the algebra Baire$([0,1]^\kappa)$, which coincide with Borel$([0,1]^\kappa)$ when κ is countable. To give more uniform notations, in this work we follow [5] and therefore $IRL(\kappa)$ will denote the free κ-generated algebra, where κ is an arbitrary cardinal. The elements of $IRL(\kappa)$ will be called *IRL-polynomials*.

If I, S and P denote the standard universal-algebraic operators in $\mathbf{RMV}_\sigma$, algebras that belong to $SP([0,1])$ will be called Riesz tribes (or simply tribes), while algebras in $ISP([0,1])$ have been characterized in [5] as follows.

Theorem 2.1. *Let $A \in \mathbf{RMV}_\sigma$. The following are equivalent:*

(i) $A \in ISP([0,1])$.

(ii) *The intersection of all MV-maximal σ-ideals of A is trivial, in symbols* $\bigcap\{M \mid M \in \mathcal{M}_\sigma(A)\} = \{0\}$.

(iii) *There exist a cardinal κ and a set V, that is an arbitrary intersection of elements of $\mathcal{BA}([0,1]^\kappa)$, such that $A \simeq IRL(\kappa)|_V$, the algebra of restrictions to V of elements of $IRL(\kappa)$.*

We call *σ-semisimple* any algebra that satisfies one of the equivalent condition of Theorem 2.1. They will be the protagonists of this paper. Moreover, an *IRL-algebraic variety* is any arbitrary intersection of elements of $\mathcal{BA}([0,1]^\kappa)$.

Note that, if $A \simeq IRL(\kappa)|_V$, by [16, Lemma 4.6] it follows that A is the algebra of all $\mathcal{BA}(V)$-measurable functions, where $\mathcal{BA}(V) = \{A \cap V \mid A \in \mathcal{BA}([0,1]^\kappa)\}$. We also note that by the same lemma $\mathcal{BA}(V) = \{B \subseteq V \mid \chi_B \in IRL(\kappa)|_V\}$.

Furthermore, we will use the following operators. For any subset $S \subseteq [0,1]^\kappa$,

$$\mathbb{I}(S) = \{p \in IRL(\kappa) \mid p(\mathbf{x}) = 0 \text{ for any } \mathbf{x} \in S\}.$$

Given a set J of IRL-polynomials,

$$\mathbb{V}(J) = \{\mathbf{x} \in [0,1]^\kappa \mid p(\mathbf{x}) = 0 \text{ for any } p \in J\} = \bigcap_{p \in J} \mathbb{V}(\{p\}).$$

These operators induce a categorical duality between σ-semisimple algebras and IRL-algebraic varieties, see [5, Section 4] and with this notations, if $A \simeq$

$IRL(\kappa)/J$ is a presentation of the algebra A, then $A \simeq IRL(\kappa)|_{\mathbb{V}(J)}$.

Finally, we consider the topology $\mathcal{ZIRL}$ on $[0,1]^\kappa$ that is generated by elements of $\mathcal{BA}([0,1]^\kappa)$ taking them as closed subsets, rather than using the more familiar open-convention. Thus, a closed in $([0,1]^\kappa, \mathcal{ZIRL})$ is an arbitrary intersection of Baire sets, and Baire sets are clopens in $\mathcal{ZIRL}$. With the previous notations, it holds that V is closed in $\mathcal{ZIRL}$ if, and only if, $V = \bigcap_{p \in F} \mathbb{V}(p)$, where $F \subseteq IRL(\kappa)$ is an arbitrary set.

This topology is the one given in [5, Section 5] where it is also proved that it is not always compact. We also remark that, in the case of $([0,1], \mathcal{ZIRL})$ the topology is generated by $\mathcal{BA}([0,1]) = \mathcal{BO}([0,1])$, which is the σ-algebra generated by intervals of type $[0,r)$ or $(s,1]$.

2.3. States and subjective probability

Probability measures are encoded in Łukasiewicz logic via the notion of a *state*, introduced by D. Mundici with the idea of obtaining an averaging process for formulas.

To define this notion properly, we recall that any MV-algebra A can be endowed with a partial operation, that we shall denote by $+$, defined when $x \odot y = 0$, for $x, y \in A$. In this case $x + y = x \oplus y$. Equivalently, if $A = \Gamma(G,u)$ via Mundici's functor, the partial sum $x + y$ is defined when $x + y \leq u$ in G.

Formally, a *state* of a σ-complete Riesz MV-algebra A is a map $s : A \to [0,1]$ satisfying the following conditions:

(1) $s(1) = 1$,

(2) for all $x, y \in A$ such that $x \odot y = 0$, $s(x \oplus y) = s(x) + s(y)$.

A σ-state is a state that, in addition, preserves countable suprema of increasing sequences, that is,

(3) If $\{a_n\}_{n \in \mathbb{N}}$ is an increasing sequence of elements of A, then $s(\bigvee_n a_n) = \bigvee_n s(a_n)$.

Following [8] we denote by $\mathcal{S}(A)$ the set of states of any MV-algebra A. It is known, see [8, Theorem 4.1.1] that $\mathcal{S}(A) = \overline{co}(Hom(A,[0,1]))$, where $Hom(A,[0,1]) = \{h : A \to [0,1] \mid h \text{ is a homomorphism}\}$ and $\overline{co}$ denotes the topological closure (in the product topology of the euclidean space) of the convex hull of $Hom(A,[0,1])$. Thus, any state is either a homomorphism, a convex combination of homomorphisms, a limit of a net of convex combinations. We will use the same notation for σ-complete Riesz MV-algebras, while $\mathcal{S}_\sigma(A)$ will denote the set of σ-states of A.

In the case of tribes, we can obtain an integral representation for σ-states, which was firstly proved by Butnariu and Klement, see [1, Chapter II, Section 6].

Theorem 2.2. *For every Riesz tribe $\mathcal{T} \subseteq [0,1]^X$, for every σ-state s of $\mathcal{T}$, for every $f \in \mathcal{T}$,*

$$s(f) = \int_X f \, \mathrm{d}\mu_s.$$

The measure $\mu_s : \mathcal{S}(\mathcal{T}) \to [0,1]$ is given by $\mu(A) = s(\chi_A)$ and $\mathcal{S}(\mathcal{T}) = \{A \subseteq X \mid \chi_A \in \mathcal{T}\}$.

Note that a more general version of Theorem 2.2 is the so-called Kroupa-Panti theorem, see [8, Theorem 4.0.1] for a precise statement.

If $\mathcal{T}$ is a Riesz tribe, it follows from [21, Lemma 11.8] that $\mathcal{T}$ is closed under the usual product of functions, that is, if $f, g \in \mathcal{T}$, then $f \cdot g \in \mathcal{T}$. Those MV-algebras that can be endowed with a ring-like structure are known and studied in literature under the name of PMV-algebras, see [19]. Thus, a consequence of these remarks is the fact that any Riesz tribe has a natural structure of PMV-algebra. We also note that states on a PMV-algebra P are defined as states on the MV-reduct of P, see [8, Section 7]. Finally, in the sequel we shall use the fact that for any $f, g \in \mathcal{T}$, if $\mathbf{1}$ is the function identically equal to 1, by [19, Lemma 2.9], $f \cdot g \leq f \cdot \mathbf{1} = f$ and $f \cdot g \leq \mathbf{1} \cdot g = g$. Moreover, we recall that states are monotone functions: $f \leq g$ implies $s(f) \leq s(g)$.

A *probability Riesz tribe* is a pair $(\mathcal{T}, s)$ where $\mathcal{T}$ is a Riesz tribe and s is a σ-state. A κ-dimensional observable is defined as a homomorphism $\mathcal{X} \colon IRL(\kappa) \to \mathcal{T}$, see [4], and it is essentially the algebraic counterpart of a random variable on $\mathcal{T} \subseteq [0,1]^X$. Indeed, following [4, 5] to any observable $\mathcal{X}$ posed in a Riesz probability tribe it corresponds a unique random variable $f \colon (X, \mathcal{S}(\mathcal{T}), \mu_s) \to ([0,1]^\kappa, \mathcal{BA}([0,1]^\kappa))$, with the same notations given before, such that $\mathcal{X}(a) = a \circ f$ for any $a \in IRL(\kappa)$.

Finally, κ-dimensional stochastic processes posed in $\mathcal{T}$ are defined as sequences of κ-dimensional observables $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ with values in the same Riesz tribe. If $\mathcal{T}$ carries a σ-state s and if $\mathcal{X}_n = - \circ f_n$, the process is called *weakly exchangeable* if the sequence of classical random variables $\{f_n\}_{n \in \mathbb{N}}$ is exchangeable, see [16]. That is, if for each $n \in \mathbb{N}$ the joint distribution of two finite subsets $f_{i_1}, \ldots, f_{i_n}$ and $f_{j_1}, \ldots, f_{j_n}$ is the same. In symbols, if for $A_1, \ldots, A_n \in \mathcal{BA}([0,1]^\kappa)$,

$$\mu_s\left(f_{i_1} \in A_1, \ldots, f_{i_n} \in A_n\right) = \mu_s\left(f_{j_1} \in A_1, \ldots, f_{j_n} \in A_n\right).$$

Remark 2.3. To avoid any confusion, we shall call *σ-homomorphisms* the arrows of $\mathbf{RMV}_\sigma$, while we will use the term *homomorphisms* for the arrows in the algebraic category of Riesz MV-algebras, containing $\mathbf{RMV}_\sigma$. We will denote the sets of σ-homomorphisms and homomorphisms from A to B by $Hom_\sigma(A, B)$ and $Hom(A, B)$ respectively. Moreover, we will call σ-function any function that preserves the supremum of any increasing sequence.

3. The structure of the spaces of σ-homomorphisms and σ-states

de Finetti coherence criterion has been extended to Łukasiewicz logic by D. Mundici, see [8, 21] for an up-to-date formulation of the statement. In these references, one key point towards coherence is the characterization of the state-space of an MV-algebra A as a compact convex subset of $[0,1]^A$ endowed with the product Euclidean topology. Indeed, this fact allows to use the well-known Krein-Milman theorem and obtain coherence from the very general proof given in [15]. In this section we see how the countable operation breaks compactness of the space of σ-homomorphisms.

Lemma 3.1. *Let $Form(\kappa)$ be the set of formulas in $\mathcal{IRL}$ build upon κ propositional variables, where κ is an arbitrary cardinal. Then:*

1. *There is a one-one correspondence between evaluations $v : Form(\kappa) \to [0,1]$ and σ-homomorphisms of $IRL(\kappa)$ in $[0,1]$.*

2. *There is a one-one correspondence between points of $[0,1]^\kappa$ and elements of $Hom_\sigma(IRL(\kappa),[0,1])$.*

3. *There is a homeomorphism between $([0,1]^\kappa, \mathcal{ZIRL})$ and $\mathcal{M}_\sigma(IRL(\kappa))$ endowed with the hull-kernel topology.*

Proof. The claims are easily deduced from computation. The correspondences are given by the following stipulations:

1. Take $\varphi \in Form(\kappa)$ and let $p \in IRL(\kappa)$ be the function that uniquely corresponds to φ. To each evaluation v we can associate the homomorphism h_v given by $h_v(p) = v(\varphi)$, which is well defined by the definition of the Lindenbaum-Tarski algebra of $\mathcal{IRL}$. Conversely, to each homomorphism h we associate v_h given by $v_h(\varphi) = h(p)$. See [3, Lemma 4.5.6] for further details in the case of Łukasiewic logic.

2. To each point $\mathbf{x} \in [0,1]^\kappa$ we associate the homomorphism $h_{\mathbf{x}} : IRL(\kappa) \to [0,1]$ given by the evaluation $h_{\mathbf{x}}(p) = p(\mathbf{x})$. Conversely, to each homomorphism $h : IRL(\kappa) \to [0,1]$ we associate the point $\mathbf{x}_h = (h(\pi_i)_{i\in\kappa})$. The correspondence is one-one because the projections π_i are generators for $IRL(\kappa)$ and each $p \in IRL(\kappa)$ is therefore an IRL-combination of the projections.

3. It follows from [5, Lemma 4.6] that the function $\eta\colon [0,1]^\kappa \to \mathcal{M}_\sigma(IRL(\kappa))$ given by $\mathbf{x} \mapsto \mathbb{I}(\mathbf{x})$ is a bijection. The proof that η is a homeomorphism is standard. Let $F_a = \{M \in \mathcal{M}_\sigma(IRL(\kappa)) \mid a \in M\}$ be a basic closed in $\mathcal{M}_\sigma(IRL(\kappa))$. To prove that η is continuous, we notice that $\eta^{-1}(F_a) = \mathbb{V}(a)$, while to prove that it is closed, take any closed in $\mathcal{ZIRL}$, that is, $F = \bigcap_{a\in I} \mathbb{V}(a) = \mathbb{V}(I)$. It is easy to see that $\eta(F) = \{M \in \mathcal{M}_\sigma(IRL(\kappa)) \mid I \subseteq M\}$, which is a closed set in the hull-kernel topology (relative to σ-ideals).

□

It is important to remark that, when dealing with MV-algebras without the infinitary operation $\bigvee$, $Hom(A,[0,1])$ and $Max(A)$ are homeomorphic topological spaces, where $Max(A)$ is endowed with the usual hull-kernel topology, while

$Hom(A,[0,1])$ is endowed with the topology induced by the product Euclidean topology on $[0,1]^A$. In this case, one can see that $Hom(A,[0,1])$ is closed in $[0,1]^A$. This latter fact has an implicit assumption: the Łukasiewicz operations are continuous with respect to the euclidean topology, which coincide with the Zariski topology obtained starting from the free MV-algebra.

In contrast to the previous considerations, the countable disjunction is not continuous with respect to the Euclidean topology.

Lemma 3.2. *All operations of σ-complete Riesz MV-algebra are continuous with respect to* $([0,1],\mathcal{ZIRL})$. *In particular, the countable operation* $\bigvee : [0,1]^\omega \to [0,1]$ *is not continuous with respect to the product of the euclidean topology, while it is continuous with respect to* $\mathcal{ZIRL}$.

Proof. The operations $\oplus$, $\neg$ and the scalar multiplication are continuous with respect to the Euclidean topology on $[0,1]$ and its product topology. This means that any of these operations, let say $f \in \{\oplus, \neg, \alpha\}$, is Baire-measurable, see [5, Remark 2.1]. Thus, for any Baire set $B \in \mathcal{BA}([0,1])$, $f^{-1}(B) \in \mathcal{BA}([0,1]^n)$, for the appropriate $n = 1,2$. Since Baire sets generate the topology $\mathcal{ZIRL}$ on all $[0,1]^n$, f is continuous with respect to $\mathcal{ZIRL}$.

For the countable suprema, take $[0,r)$ to be an open set in $[0,1]$ with the Euclidean topology. Then, $(a_n)_{n\in\mathbb{N}} \in \bigvee^{-1}([0,r))$ if, and only if, there exists $q \in [0,1] \cap \mathbb{Q}$ such that $a_n \leq q < r$ for any n. Thus, $\bigvee^{-1}([0,r)) = [0,q]^\omega$, which is not an open in the product euclidean topology.

On the other end, since intervals of type $[0,r)$ are generators for $\mathcal{BA}([0,1]) = \mathcal{BO}([0,1])$, for any $[0,r)$ we have that $\bigvee^{-1}([0,r)) = [0,q]^\omega$ is a Baire subset of $[0,1]^\omega$ endowed with the product σ-algebra $\mathcal{BA}([0,1]^\omega) = \bigtimes_\omega \mathcal{BA}([0,1])$, see [10, Section 38, Exercise (4)]. Consequently, $\bigvee$ is Baire measurable and it is continuous with respect to $\mathcal{ZIRL}$. □

Notice that, looking at $[0,1]^\kappa$ as a topological space, there are at least two ways of endowing it the the Zariski IRL-topology:

(i) We take the product of $([0,1],\mathcal{ZIRL})$,

(ii) We take the topology $\mathcal{ZIRL}$ on $[0,1]^\kappa$, that is, the topology generated by the zerosets of Baire functions in $IRL(\kappa)$.

In Lemma 3.2 we have used the topology of (ii). Nonetheless, we shall now see that in our case they coincide. To do so, we need a preliminary lemma.

Lemma 3.3. *The topology* $([0,1],\mathcal{ZIRL})$ *is Hausdorff.*

Proof. This is because the MV-Zariski topology on $[0,1]$, that is Hausdorff, is smaller than $\mathcal{ZIRL}$. Indeed, the MV-Zariski topology is generated by the zerosets of the free one-generated MV-algebra, which is contained in $IRL(1)$ and therefore are among the generators of the topology $\mathcal{ZIRL}$. □

Proposition 3.4. *The topologies on* $[0,1]^\kappa$ *defined in items (i) and (ii) above coincide. Whence, we shall call it* product Zariski topology.

Proof. By Lemma 3.3 and [2, Lemma 6.2] it is enough to prove that any *definable function* $t : [0,1]^\kappa \to [0,1]$ is continuous with respect to $[0,1]^\kappa$ endowed with the product topology of the Zariski topology (the topology of item (i) above), and $[0,1]$ endowed with the Zariski topology. Notice that, in the setting of $\mathbf{RMV}_\sigma$, definable functions are elements of $IRL(\kappa)$. Whence, we prove the claim by structural induction on the construction of the terms using half-open intervals of $[0,1]$.

If $t = \pi_i$ is one of the projections, the claim follows from the definition of the product topology, that makes all projections continuous.

For the negation and scalar operations, the inductive step is straightforward.

If $t = p \oplus q$, for every $a \in [0,1]$, we have $(p \oplus q)(\mathbf{x}) < a$ if, and only if, there is a rational $r < a$ such that $p(\mathbf{x}) < r$ and $q(\mathbf{x}) < a - r$, thus, $t^{-1}([0,a)) = p^{-1}([0,r)) \cap q^{-1}([0,a-r))$, which are both closed in the product topology by induction hypothesis and the fact that $[0,r)$ and $[0,a-r)$ are Baire sets, that is, basic closed of $([0,1], \mathcal{ZIRL})$.

If $t = \bigvee_n p_n$, we have $\bigvee_n p_n(\mathbf{x}) < a$ if, and only if, there is a rational $r < a$ such that $p_n(\mathbf{x}) \leq r$ for every n. Thus, $t^{-1}([0,a)) = \bigcap_n \left(p_n^{-1}([0,r])\right)$, which is a closed subset of $[0,1]^\kappa$ by induction hypothesis and the fact that $[0,r]$ is a Baire subset of $[0,1]$. $\square$

The argument of the following proof is mostly standard, we give details for the sake of completeness.

Lemma 3.5. *For any $A \in \mathbf{RMV}_\sigma$, $\mathcal{M}_\sigma(A)$ and $Hom_\sigma(A,[0,1])$ are homeomorphic, with the hull-kernel topology on the former and the product Zariski topology on the latter.*

Proof. Consider the map $\eta\colon Hom_\sigma(A,[0,1]) \to \mathcal{M}_\sigma(A)$, sending $h \mapsto ker(h)$. The map is surjective, indeed we recall that for any $M \in \mathcal{M}_\sigma(A)$, A/M is a simple and σ-complete Riesz MV-algebra, and therefore it is isomorphic to $[0,1]$. Thus, $h : A \to A/M \to [0,1]$ is a σ-homomorphisms and $ker(h) = M$. Injectivity is easily checked by direct computation, and it also follows from the fact that η is the restriction and co-restriction of the analogous map defined on the MV-reduct of A, see [21, Theorem 4.16].

Let us prove that η is a homeomorphism. Take $F_a = \{M \in \mathcal{M}_\sigma(A) \mid a \in M\}$, which is a basic closed for the hull-kernel topology. Then

$$\eta^{-1}(F_a) = \{h \in Hom_\sigma(A,[0,1]) \mid h(a) = 0\} = \pi_a^{-1}(\{0\}) \cap Hom_\sigma(A,[0,1]).$$

Since $\pi_a\colon [0,1]^A \to [0,1]$ is continuous with respect to the product Zariski topology on $[0,1]^A$ and the Zariski topology on $[0,1]$, the set $\pi_a^{-1}(\{0\}) \cap Hom_\sigma(A,[0,1])$ is closed and η is continuous.

Let us prove that η is closed, or equivalently, that η^{-1} is continuous. Let $U = \prod_{a \in A} U_a$ be an basic closed in the product topology. Let $\{a_1, \ldots, a_k\} \subseteq A$ the set of elements such that $U_a \neq [0,1]$, assume without loss of generality that they are basic closed sets of $([0,1], \mathcal{ZIRL})$ and let $\{p_1, \ldots p_k\} \subseteq IRL(1)$ be the set of functions such that $U_{a_i} = \mathbb{V}(p_i)$. Thus,

$$h \in U \Leftrightarrow h(a_i) \in \mathbb{V}(p_i) \text{ for any } i = 1, \ldots, k \Leftrightarrow p_i(h(a_i)) = 0 \text{ for any } i = 1, \ldots, k.$$

Since $IRL(1)$ is the free one-generated algebra in $\mathbf{RMV}_\sigma$, it is also isomorphic to the algebra of term functions $f_\tau : [0,1] \to [0,1]$. Thus, if τ_i is the term that corresponds to p_i, we have that $p_i(h(a)) = h(\tau_i(a))$. Therefore,

$$h \in U \Leftrightarrow b_i := \tau_i(a) \in ker(h) = \eta(h) \quad \text{for any} \quad i = 1, \dots, k.$$

Whence, $\eta(U) = \{M \in \mathcal{M}_\sigma(A) \mid M = ker(h), h \in U\} = \{M \in \mathcal{M}_\sigma(A) \mid b_i \in M,\ i = 1, \dots, k\} = \bigcap_{i=1}^k F_{b_i}$, which is closed in the hull-kernel topology of $\mathcal{M}_\sigma(A)$. □

Proposition 3.6. *$Hom_\sigma(A, [0,1])$ is closed in $[0,1]^A$ with the product Zariski topology.*

Proof. $Hom_\sigma(A, [0,1])$ is the intersection, in $[0,1]^A$, of the subsets

$$\begin{aligned}
&S_1 = \{f \in [0,1]^A \mid f(a \oplus b) = f(a) \oplus f(b)\} \\
&S_2 = \{f \in [0,1]^A \mid f(\neg a) = 1 - f(a)\} \\
&S_3 = \{f \in [0,1]^A \mid f(1_A) = 1\} \\
&S_4 = \left\{ f \in [0,1]^A \mid f\left(\bigvee_n a_n\right) = \bigvee_n f(a_n) \right\}.
\end{aligned}$$

By Lemmas 3.2 and 3.3, all of these sets are closed in $([0,1]^A, \mathcal{ZIRL})$, and therefore $Hom_\sigma(A, [0,1])$ is closed. □

As mentioned in Section 2.3, for any MV-algebra A the state-space $\mathcal{S}(A)$ is a convex compact subset of $[0,1]^A$, where the topology is induced by the topology on $Hom(A, [0,1])$. The notion of convergence on states is the weak*-convergence defined by $s_\gamma \to s$ if $s_\gamma(a) \to s(a)$ for any $a \in A$, see [8, Equation (3) and Theorem 4.0.1] for the case of MV-algebras. We will prove an analogous result on σ-states.

Proposition 3.7. *The space of σ-states of $A \in \mathbf{RMV}_\sigma$, denoted by $\mathcal{S}_\sigma(A)$, is a convex subset of $[0,1]^A$.*

Proof. If $s = \alpha_1 s_1 + \dots + \alpha_k s_k$, with $\alpha_i \geq 0$, $\sum_{i=1}^k \alpha_i = 1$ and $s_i \in \mathcal{S}_\sigma(A)$, the claim follows from the distributivity of $\oplus$ over $\bigvee$, the remark that the sequence $\{a_n\}_{n \in \mathbb{N}}$ is increasing, the monotonicity of states and the fact that, being a convex combination, the partial sum $+$ is always well defined and it coincides with $\oplus$. Indeed, let $\{a_n\}_{n \in \mathbb{N}}$ be an increasing sequence. We have

$$\begin{aligned}
s\left(\bigvee_n a_n\right) &= \alpha_1 s_1 \left(\bigvee_n a_n\right) + \dots + \alpha_k s_k \left(\bigvee_n a_n\right) \\
&= \bigvee_n (\alpha_1 s_1(a_n)) + \dots + \bigvee_n (\alpha_k s_k(a_n)) \\
&= \bigvee_{i_1 \in I_1} \cdots \bigvee_{i_k \in I_k} (\alpha_1 s_1(a_{i_1}) + \ldots + \alpha_k s_k(a_{i_k})),
\end{aligned}$$

where $I_1, \dots I_k$ are countable sets used to relabel the indexes of the sequence. Since $\{a_n\}_{n\in\mathbb{N}}$ is increasing, and since each s_i is monotone, each sum $\alpha_1 s_1(a_{i_1}) + \dots + \alpha_k s_k(a_{i_k})$ is dominated by the sum $\alpha_1 s_1(a_n) + \dots + \alpha_k s_k(a_n)$, with n such that $a_n = \sup\{a_{i_1}, \dots, a_{i_k}\}$. Thus, in doing the supremum on the right side we can replace the "mixed" terms with convex combinations evaluated on the same element of the sequence. Therefore,

$$s\left(\bigvee_n a_n\right) = \bigvee_n (\alpha_1 s_1(a_n) + \dots + \alpha_k s_k(a_n)) = \bigvee_n s(a_n).$$

□

Proposition 3.8. *The elements of $Hom_\sigma(A, [0,1])$ are the extreme points in the convex $\mathcal{S}_\sigma(A)$.*

Proof. It follows from the fact that the elements of $Hom(A, [0,1])$ are the extreme points in the convex $\mathcal{S}(A)$. Indeed, if $h \in Hom_\sigma(A, [0,1])$ is not extremal in $\mathcal{S}_\sigma(A)$, there are $s_1, s_2 \in \mathcal{S}_\sigma(A)$ and $\alpha \in [0,1]$ such that $\alpha s_1 + (1-\alpha)s_2 = h$. Thus, in particular, h won't be extremal in $\mathcal{S}(A)$, a contradiction. □

The following proposition and lemma are of independent interest and we record them here for future use. By *additive function* we mean a function that preserves the partial sum $+$ defined in Section 2.3.

Proposition 3.9. *Let $A \in \mathbf{RMV}_\sigma$ and let $f : A \to [0,1]$. Then f preserves countable partial sums if, and only if, f is an additive σ-function.*

Proof. Suppose f is additive and preserves increasing countable suprema. Let $\sum_n a_n$ be a countable sum in A. Then $\sum_n a_n = \sup_n (a_1 + \dots + a_n)$ and the supremum is increasing. So

$$f\left(\sum_n a_n\right) = f(\sup_n (a_1 + \dots + a_n)) = \sup_n f(a_1 + \dots + a_n),$$

and since f is additive we have

$$\sup_n f(a_1 + \dots + a_n) = \sup_n (f(a_1) + \dots + f(a_n)) = \sum_n f(a_n).$$

By putting together the last two equations, we have that f preserves countable sums.

Conversely, suppose f preserves countable sums. First, f is additive and monotone. Indeed, for any $a, b \in A$ such that $b \leq a$, we can write $a = b \oplus (a \ominus b)$ and $b \odot (a \ominus b) = 0$. Whence, in $[0,1]$, $f(a) = f(b) + f(a \ominus b)$. Moreover, $f(a \ominus b) = f(a) - f(b)$.

Let $\sup_n a_n$ be a countable increasing supremum. Then, we can write $a_n = a_1 \oplus \sum_{i=1}^{n-1} (a_{i+1} \ominus a_i)$ and therefore $f(a_n) = f(a_1) + \sum_{i=1}^{n-1} (f(a_{i+1}) - f(a_i))$.

Consequently, by taking the supremum over n,

$$\sup_n a_n = a_1 \oplus \sum_{i\in\omega}(a_{i+1} \ominus a_i) \qquad \text{and}$$
$$\sup_n f(a_n) = f(a_1) + \sum_{i\in\omega}\left(f(a_{i+1}) - f(a_i)\right),$$

hence, by applying f to $\sup_n a_n$, we have

$$f(\sup_n a_n) = f\left(a_1 \oplus \sum_{i\in\omega}(a_{i+1} \ominus a_i)\right) = f\left(a_1\right) + \sum_{i\in\omega} f\left(a_{i+1} \ominus a_i\right)$$
$$= f(a_1) + \sum_{i\in\omega}\left(f(a_{i+1}) - f(a_i)\right) = \sup_n f(a_n).$$

Thus, f preserves increasing countable suprema. $\square$

Lemma 3.10. *Let A be a σ-complete Riesz MV-algebra, let s be a state and let $\{s_k\}_{k\in K}$ be a net of σ-states in $[0,1]^A$. If $s = sup_k s_k$, then s is a σ-state.*

Proof. For any increasing sequence $\{a_n\}_{n\in\mathbb{N}}$, the inequality

$$\bigvee_n s(a_n) \leq s\left(\bigvee_n a_n\right)$$

follows by monotonicity.

Conversely, since $s_k \leq s$,

$$s_k\left(\bigvee_n a_n\right) = \bigvee_n s_k(a_n) \leq \bigvee_n s(a_n).$$

Consequently, passing to the supremum over k,

$$s\left(\bigvee_n a_n\right) \leq \bigvee_n s(a_n),$$

settling the claim. $\square$

We recall that a G_δ subset of $[0,1]^\kappa$ is a countable intersection of open sets and it belongs to $\mathcal{BO}([0,1]^\kappa) = \mathcal{BA}([0,1]^\kappa)$ for $\kappa \leq \omega$. Notice that here $[0,1]^\kappa$ needs to be endowed with the standard Euclidean topology, which is the one used to define Baire and Borel subsets.

Proposition 3.11. *Let $\kappa \leq \omega$, let $V \subseteq [0,1]^\kappa$ be a G_δ subset with respect to the Euclidean topology. Any state $s : IRL(\kappa)|_V \to [0,1]$ is a σ-state.*

Proof. By Mundici's equivalence and [21, Proposition 10.3], any $s \in \mathcal{S}(IRL(\kappa))$ extends to a positive liner functional $t\colon \mathrm{Baire}(V) \to \mathbb{R}$, where $\mathrm{Baire}(V)$ is the Banach lattice of bounded Baire-measurable functions in $\mathbb{R}^V$. Such a V is a Polish space (separable, completely metrizable space) since the cube $[0,1]^\kappa$ is a Polish space itself and G_δ subsets of Polish spaces are Polish by Alexandrov Theorem, see [14, Chapter 3, Section 33, Subsection VI]. Consequently, by [9, Example 3.10(e)], $\mathrm{Baire}(V)$ has the *σ-order continuity property.* This property implies that for any decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathrm{Baire}(V)$, if $a_n \downarrow a$ then $t(a_n) \downarrow t(a)$. From the latter, it is easily deduced that s is a σ-state. □

As mentioned in the proof of Proposition 3.11, any V that satisfies the hypothesis is a Polish space, that is, a separable, completely metrizable space. Conversely, it is known that any Polish space is homeomorphic to a G_δ subset of the Hilbert cube. These spaces are of among the most used in probability theory, making the previous proposition quite relevant for our research project.

Thus, let us describe the state-space of σ-semisimple algebras "presented" by G_δ subsets of hypercubes.

Proposition 3.12. *Let $V \subseteq [0,1]^\kappa$ a G_δ subset, with $\kappa \leq \omega$, and take $A \simeq IRL(\kappa)|_V$. Then $\mathcal{S}(A) = \mathcal{S}_\sigma(A)$ and it is a closed subset of* $([0,1]^A, \mathcal{ZIRL})$.

Proof. The first part of the claim was proved in Proposition 3.11. To see that $\mathcal{S}_\sigma(A)$ is closed we prove that the limit of any convergent net of states is a state. Whence, let $s = \lim_\gamma s_\gamma$, $s_\gamma \in \mathcal{S}_\sigma(A)$ for any index γ. Using Lemma 3.2, it is easy to see that $s(a+b) = s(a) + s(b)$ and $s(1_A) = 1$, using the definition of convergence, the fact that each s_γ is a state, and the fact that for any $a \in A$, $(s_\gamma(a))_{\gamma\in\Gamma}$ is a net in $([0,1], \mathcal{ZIRL})$. □

We end this section with some comments. We have seen, following Lemmas 3.1 and 3.5 that the space $Hom_\sigma(IRL(\kappa), [0,1])$ is not always compact. Consequently, we can't directly infer that $\mathcal{S}_\sigma(A)$ is a compact convex subset of $[0,1]^A$, as it happens without the countable operation. Nevertheless, we proved that $\mathcal{S}_\sigma(A)$ is closed and convex on certain σ-semisimple algebras. As a deeper topological analysis of the space of σ-states goes beyond the scope of this paper, we leave the analysis of its compacteness as an open problem.

4. Dutch-book arguments in $\mathbf{RMV}_\sigma$

In [8, Section 5] one can find a detailed account of the recent literature on de Finetti's coherence criterion for many-valued events. The survay includes the work [20] in which F. Montagna extends Mundici's version of de Finetti coherence criterion, adding the possibility to consider *conditional events.*

We note that, in order to discuss conditioning, one has the need to consider MV-algebras enriched by real constants and a ring-like product. Indeed, Montagna formally defines coherence using the evaluation of the conditioning event as a *scaling factor* in the setting of *PMV-algebras.*

As remarked in Section 2.3, Riesz tribes carry a natural structure of PMV-algebra and they contain all constant functions. Whence, given these considerations, de Finetti's coherence criterion translates in our setting as follows.

Definition 4.1. A conditional event in a σ-semisimple algebra A is a pair $(p, q) \in A \times A$. Given a finite set

$$E = \{(p_1, q_1), \ldots, (p_n, q_n), r_1, \ldots, r_m\}$$

where all p_i, q_i, r_j belong to A for any $i = 1, \ldots, n$ and $j = 1, \ldots, m$, a *conditional book* on the conditional and unconditional events of E is the assignment

$$\beta \colon (p_1, q_1) \mapsto \alpha_1, \ldots, (p_n, q_n) \mapsto \alpha_n, r_1 \mapsto c_1, \ldots, r_m \mapsto c_m \tag{CB}$$

where $\alpha_i, c_j \in [0, 1]$ for all indexes.

A book is said *complete* if for any q_i there exists a unique index j such that $q_i = r_j$, that is $\{q_1, \ldots q_n\} \subseteq \{r_1, \ldots, r_m\}$ with no repetitions. In this case, we shall always assume that the events are ordered in such a way that $q_i = r_i$ for any $i = 1, \ldots, n$. A complete book is said *positive* if $c_i > 0$ for any $i = 1, \ldots, n$.

In this section κ is always assumed to be countable.

Definition 4.2. Let $A \simeq IRL(\kappa)|_V$ be a σ-semisimple algebra. The book β is *conditionally coherent* if, and only if, for any $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$ and $\delta_1, \ldots, \delta_m \in \mathbb{R}$ there exists $\mathbf{x} \in V$ such that

$$\sum_{t=1}^{n} \sigma_t q_t(\mathbf{x})(\alpha_t - p_t(\mathbf{x})) + \sum_{j=1}^{m} \delta_j (c_j - r_j(\mathbf{x})) \geq 0 \tag{CC}$$

If $E = \{r_1, \ldots, r_m\}$, that is, there are no conditional events, the book will be also called *unconditional* and, eventually, *coherent* instead of conditionally coherent.

Note that each p_i can be regarded as a formula φ_i in the logic $\mathcal{IRL}$ in which only a finite number of propositional variables occur. Moreover, $p_i(\mathbf{x})$ can be regarded as the evaluation of the formula φ_i induced by the assignment of Lemma 3.1. Obviously, the same remarks hold true for each q_i and each r_j.

Moreover, when $A \simeq IRL(\kappa)|_V$, for any $h \in Hom_\sigma(A, [0, 1])$ there exists a unique $\mathbf{x} \in V$ such that $h(a_i) = a_i(\mathbf{x})$, exactly as sketched in Lemma 3.1. Here we remark that $\{\pi_i|_V\}_{i \in \kappa}$ is again a set of generators for $IRL(\kappa)|_V$.

Therefore, Definition 4.2 can be generalized to the following version of de Finetti's coherence.

Definition 4.3. Let $A \in \mathbf{RMV}_\sigma$ and $E = \{(a_1, b_1), \ldots, (a_n, b_n), d_1, \ldots, d_m\}$. The book β is *conditionally coherent* if, and only if, for any $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$ and $\delta_1, \ldots, \delta_m \in \mathbb{R}$ there exists $h \in Hom_\sigma(A, [0, 1])$ such that

$$\sum_{t=1}^{n} \sigma_t h(b_t)(\alpha_t - h(a_t)) + \sum_{j=1}^{m} \delta_j (c_j - h(d_j)) \geq 0.$$

Definition 4.3 makes more clear the fact that in this setting we need an approach to coherence that is different from the one usually given in the setting of MV-algebras: the space $Hom_\sigma(A,[0,1])$ might not be compact, as always happens with $Hom(A,[0,1])$, and compacteness is crucial for applying the results of [15]. Whence, in our infinitary setting, we shall give a proof that is inspired by the relation between the Hahn-Banach theorem and de Finetti coherence that can be found in [22]. To ease our way in the proof, we shall first consider the case of an unconditional book.

We recall that a sublinear functional between Riesz spaces is a map $p: L \to M$ such that $p(f+g) \leq p(f) + p(g)$ for all $f, g \in L$ and $p(af) = ap(f)$ for all real numbers $a \geq 0$. We also urge the reader to consult [25, Theorem 83.13] for the version of the Hahn-Banach theorem most suited to our framework.

Theorem 4.4. *For any G_δ subset V of $[0,1]^\kappa$, let $E = \{p_1, \ldots, p_n\} \subseteq IRL(\kappa)|_V$ be a finite subset of events. Consider an assignment $\beta\colon E \to [0,1]$. Then the following are equivalent:*

(i) β is coherent*,*

(ii) β can be extended to a σ-state on $IRL(\kappa)|_V$.

Proof. Let $\mathrm{Baire}(V) \subseteq \mathbb{R}^V$ denote the set of all bounded Baire functions restricted to V. We have that $IRL(\kappa)|_V = \Gamma(\mathrm{Baire}(V), \mathbf{1})$ and $E \subseteq \mathrm{Baire}(V)$.

(i) $\Rightarrow$ (ii) The coherence of β implies that for any $\sigma_1, \ldots \sigma_n \in \mathbb{R}$,

$$\sup_{\mathbf{x} \in V} \left(\sum_{i=1}^{n} \sigma_i(\beta(p_i) - p_i(\mathbf{x})) \right) \geq 0$$

Take $q_i := \beta(p_i) - p_i$ and let $E' \subseteq \mathrm{Baire}(V)$ be the linear space generated by $q_1, \ldots, q_n$. The supremum is a sublinear functional on $\mathrm{Baire}(V)$ and for any $g \in E'$, $0 \leq \sup_{\mathbf{x} \in V} g(x)$. Thus, there exists a liner functional $t: \mathrm{Baire}(V) \to \mathbb{R}$ that extends the zero functional $\mathbf{0}$ and it is bounded by the supremum functional. Moreover, such a t preserves constant functions and it is positive, indeed we have

$$\sup_{\mathbf{x} \in V}(g(\mathbf{x})) \geq t(g) = -t(-g) \geq -\sup_{\mathbf{x} \in V}(-g(\mathbf{x})) = \inf_{\mathbf{x} \in V}(g(\mathbf{x})).$$

Thus, the restriction $s: IRL(\kappa)|_V \to [0,1]$ is a state of the Riesz MV-algebra $IRL(\kappa)|_V$ and $t(q_i) = 0$ implies that $t(p_i) = s(p_i) = \beta(p_i)$. Finally, by Proposition 3.11, s is a σ-state.

(ii) $\Rightarrow$ (i) Assume that β can be extended to a σ-state s, and let $\mu_s: \mathcal{BA}(V) \to [0,1]$ be the unique σ-additive measure associated to s via the Butniaru-Klement integral representation. Note that each $p_i \in E$ is μ_s-integrable. Let $g \in \mathrm{Baire}(V)$ denote an arbitrary payoff function $g = \sum_{i=1}^{n} \sigma_i(\beta(p_i) - p_i)$. Then g is μ_s-integrable and, since $s(p_i) = \int_V p_i d\mu_s$,

$$\int_V g d\mu_s = \sum_{i=1}^{n} \sigma_i \left(\beta(p_i) - \int_V p_i d\mu_s \right) = 0 \tag{1}$$

By way of contradiction, if β is incoherent there exists a payoff function $\overline{g}$ such that $\overline{g}(\mathbf{x}) < 0$ for any $\mathbf{x} \in V$, which would imply that $\int_V \overline{g} d\mu_s < 0$ (since $\mu_s(V) > 0$) contradicting (1). Whence, for any possible choice of $\sigma_1, \ldots, \sigma_n$, there must be $\mathbf{x} \in V$ such that $g(\mathbf{x}) \geq 0$ and β is coherent. $\square$

Note that, in fact, Theorem 4.4 is a coherence criterion for those σ-semisimple Riesz MV-algebras that are "presented" by Polish spaces. Furthermore, we have obtained a countably-additive version of the criterion for a finite set of events. In the setting of Boolean algebras, the case of countable additive measures has been dealt using countable sequences of events.

We now tackle the case of conditional books. We first prove that, for *positive* complete books, we can rephrase the events and remove the conditioning altogether.

Proposition 4.5. *Let $\kappa \leq \omega$ and let $V \subseteq [0,1]^\kappa$ be a G_δ subset. With the same notations of Definition 4.2, the complete and positive book*

$$\beta\colon (p_1, q_1) \mapsto \alpha_1, \ldots, (p_n, q_n) \mapsto \alpha_n, r_1 \mapsto c_1, \ldots, r_m \mapsto c_m$$

is conditionally coherent if, and only if, the unconditional book

$$\beta^c\colon (d_1 p_1 \cdot q_1) \mapsto \frac{\alpha_1}{M}, \ldots, (d_n p_n \cdot q_n) \mapsto \frac{\alpha_n}{M}, r_1 \mapsto c_1, \ldots, r_m \mapsto c_m$$

is coherent, where $M = \max\left\{\frac{1}{c_1}, \ldots, \frac{1}{c_n}\right\}$ and $d_i = \frac{1}{Mc_i}$ for any $i = 1, \ldots, n$.

Furthermore, β is conditionally coherent if, and only if, there exists a σ-state s on $A \simeq IRL(\kappa)|_V$ such that $s(p_i \cdot q_i) = \alpha_i s(q_i)$ and $s(r_j) = c_j$, for the obvious choices of the indexes.

Proof. The proof follows from [20, Lemma 3.6 and Theorem 3.7] *mutati mutandis.* Indeed it is a matter of computation for which the main point is to choose wisely the coefficients σ_i's and δ_j's, but the exact same positions used in [8, 20] work here as well. The hypothesis on V are needed in order to apply Theorem 4.4 to β^c in the second part of the claim. $\square$

More complicated is the case of complete books in which some of the c_i's, for $i = 1, \ldots, n$, is zero. In this case, we cannot adapt the proof of [20, Lemma 3.8] because we don't have the same characterization of the state-space. Nonetheless, the measurability of our algebras gives an alternative proof.

Assume we have ordered a book β, in a way such that $c_t > 0$ for any $t = 1, \ldots, i$, while $c_t = 0$ for all $t \in \{i+1, \ldots, n\}$. Let β^i denote the restriction

$$\beta^i\colon (p_1, q_1) \mapsto \alpha_1, \ldots, (p_i, q_i) \mapsto \alpha_i, r_1 \mapsto c_1, \ldots, r_m \mapsto c_m.$$

Lemma 4.6. *Let $\kappa \leq \omega$ and let $V \subseteq [0,1]^\kappa$ be a G_δ subset. With the above notations,*

1. *β is conditionally coherent if, and only if, β^i is conditionally coherent.*

2. *There exists a σ-state s on $A \simeq IRL(\kappa)|_V$ such that for any $t = 1, \ldots, n$ and $j = 1, \ldots, m$ $s(p_t \cdot q_t) = \alpha_t s(q_t)$ and $s(r_j) = c_j$, if, and only if, there exists a σ-state s on $A \simeq IRL(\kappa)|_V$ such that for any $t = 1, \ldots, i$ and $j = 1, \ldots, m$ $s(p_t \cdot q_t) = \alpha_t s(q_t)$ and $s(r_j) = c_j$.*

Proof. 1. Trivially, if β is conditionally coherent, so is β^i. For the nontrivial direction, let $\sigma_1, \ldots, \sigma_n$ and $\delta_1, \ldots, \delta_m$ be stakes on the events in β, with the obvious choice of indexes.

Since β^i is conditionally coherent and $c_t > 0$ for all $t = 1, \ldots, i$, by Proposition 4.5 there exists a state s on $IRL(\kappa)|_V$ such that $s(p_t \cdot q_t) = \alpha_t s(q_t)$ for $t = 1, \ldots, i$ and $s(r_j) = c_j$ for $j = 1, \ldots, m$. In particular, for any $l = i+1, \ldots, n$, $s(r_l) = s(b_l) = c_l = 0$. Furthermore, by monotonicity of the state, $s(p_l \cdot q_l) \leq s(q_l) = c_l = 0$ as well.

A generic payoff function $P(\mathbf{x})$ for β can be written as

$$\sum_{t=1}^{i} \sigma_t q_t(\mathbf{x})(\alpha_t - p_t(\mathbf{x})) + \sum_{t=i+1}^{n} \sigma_t q_t(\mathbf{x})(\alpha_t - p_t(\mathbf{x})) + \sum_{j=1}^{m} \delta_j (c_j - r_j(\mathbf{x})).$$

Such a generic P is an integrable function with respect to μ_s, since it is a bounded polynomial combination of $(V, \mathcal{BA}(V), \mu_s)$-measurable functions. Therefore, we have the following:

$$\begin{aligned}
\int_V P d\mu_s = & \int_V \sum_{t=1}^{i} \sigma_t q_t (\alpha_t - p_t) d\mu_s + \int_V \sum_{t=i+1}^{n} \sigma_t q_t (\alpha_t - p_t) d\mu_s \\
& + \int_V \sum_{j=1}^{m} \delta_j (c_j - r_j) d\mu_s \\
= & \sum_{t=1}^{i} \sigma_t \left(\alpha_t \int_V q_t d\mu_s - \int_V q_t \cdot p_t d\mu_s \right) \\
& + \sum_{t=i+1}^{n} \sigma_t \left(\alpha_t \int_V q_t d\mu_s - \int_V q_t \cdot p_t d\mu_s \right) + \sum_{j=1}^{m} \delta_j \left(c_j - \int_V r_j d\mu_s \right) \\
= & \sum_{t=1}^{i} \sigma_t \left(\alpha_t s(q_t) - s(p_t \cdot q_t) \right) + \sum_{t=i+1}^{n} \sigma_t \left(\alpha_t s(q_t) - s(p_t \cdot q_t) \right) \\
& + \sum_{j=1}^{m} \delta_j \left(c_j - s(r_j) \right) = 0
\end{aligned}$$

Note that the first and third sums are zero because $\alpha_t s(q_t) = s(p_t \cdot q_t)$ for $t = 1, \ldots, i$ and $c_j = s(r_j)$ for $j = 1, \ldots, m$. The second sum is zero because for $t = i+1, \ldots, n$, we have $s(q_t) = s(q_t \cdot p_t) = 0$.

Consequently there must exist $\mathbf{x} \in V$ such that $P(\mathbf{x}) \geq 0$. Indeed, if $P < 0$ then the integral would be strictly negative as well (since $\mu_s(V) > 0$), a contradiction. Thus, β is conditionally coherent.

2. It is the same as [20, Lemma 3.8(2)]. From left-to-right, the state s (that "extends" β) restricted to β^i satisfies the claim for β^i as well. Conversely, assume that there exists a state s on $IRL(\kappa)|_V$ such that $s(p_t \cdot q_t) = \alpha_t s(q_t)$ for $t = 1, \ldots, i$ and $s(r_j) = c_j$ for $j = 1, \ldots, m$. Whence, as in item 1, for any $l = i+1, \ldots, n$, $s(q_l) = 0 = s(p_l \cdot q_l)$. Thus, the equation $\alpha_l s(q_l) = s(p_l \cdot q_l)$ is trivially satisfied for all $l = i+1, \ldots, n$ as well, settling the claim. $\square$

Finally, putting together Proposition 4.5 and Lemma 4.6, we have the following theorem.

Theorem 4.7. *The complete book*

$$\beta\colon (p_1, q_1) \mapsto \alpha_1, \ldots, (p_n, q_n) \mapsto \alpha_n, r_1 \mapsto c_1, \ldots, r_m \mapsto c_m$$

is conditionally coherent if, and only if, there exists a σ-state s on $A \simeq IRL(\kappa)|_V$ such that $s(p_i \cdot q_i) = \alpha_i s(q_i)$ and $s(r_j) = c_j$, for the obvious choices of the indexes.

We note that in the survey paper [8], Lemma 7.1.2 and Theorem 7.1.3, one can find a slightly different approach to the case of complete and positive books. In particular, positive books are defined in a slightly different way, making computation a bit more straightforward. Nonetheless, the main result – that is, Theorem 4.7 – will remain the same.

4.1. An application to logico-algebraic models

In what follows, $k \leq \omega$ is a countable cardinal and $\mathcal{C}_E$ will denote the set of all coherent books with respect to E. In [16] the authors have defined *logico-algebraic statistical models* as functions $\eta = (\eta_i)_{i\in\kappa}\colon P \to \Delta_\kappa$, where $P \subseteq [0,1]^d$ is an intersection of Baire sets and Δ_κ is the standard κ-dimensional simplex.

This definition was inspired by the theory of algebraic statistics, whose main reference is [23]. Our dictionary, that translates notions of statistics in logical terms, is the following:

- $[0,1]^n$ is the set of observations on the real world, while $IRL(n)$ is the set of events;
- the IRL-algebraic variety $P \subseteq [0,1]^d$ is the set of states of the world, or parameters. We assume that $P = \mathbb{V}(q)$ is a Baire subset of $[0,1]^d$;
- $\eta = (\eta_i)_{i\in\kappa}\colon P \to [0,1]^k$ is our statistical model: to each parameter $\mathbf{x} \in P$ it associates the tuple $(\eta_i(\mathbf{x}))_{i\in\kappa}$.

Thus, we define a *coherent model* by using de Finetti's coherence.

Definition 4.8. If $n, d, k \in \mathbb{N}$, a statistical model $\eta : P \subseteq [0,1]^d \to [0,1]^k$ is coherent with respect to the events $E = \{p_1, \ldots, p_k\} \subseteq IRL(n)$ if $\eta(P) \subseteq \mathcal{C}_E$. The elements of E therefore represent the possible events that might occur.

Notice that, in this definition, the model $\eta = (\eta_1, \ldots, \eta_k)$ is always coherent w.r.t. the events $\{\eta_1, \ldots, \eta_k\}$.

Lemma 4.9. *If $n, d, k \in \mathbb{N}$, a statistical model $\eta : P \subseteq [0,1]^d \to [0,1]^k$ is coherent with respect to the events $E = \{p_1, \ldots, p_k\} \subseteq IRL(n)$ if, and only if, for any $\mathbf{x} \in P$ there exists a state $s\colon IRL(n) \to [0,1]$ such that such that $s(p_i) = \eta_i(\mathbf{x})$ for any $i = 1, \ldots, k$.*

Proof. It follows from the definition of a coherent model and Theorem 4.4. □

Thus, when a model is coherent with respect to some set E, each point of its image can be extended to a state on $IRL(n)$. We note that Lemma 4.9 provides a justification for this logico-algebraic definition of a statistical model. Indeed, classically, statistical models are often taken to be set of probability measures.

Example 4.10. Let $k \in \mathbb{N}$ and let us consider a binomial model, which is algebraically described as the function:

$$\eta_i\colon [0,1] \to [0,1] \quad \eta_i(x) = \binom{k}{i} x^i (1-x)^{k-i}.$$

Let $\eta\colon [0,1] \to [0,1]^{k+1}$ be defined as $\eta = (\eta_0, \ldots, \eta_k)$. Here the integer k represents the iterations of an experiment, and for a fixed $x \in [0,1]$, $\eta_i(x)$ is the probability of having i successes and $k-i$ failures, given that the probability of success in one single trial is x.

Since $\sum_{i=0}^{k} \eta_i(x) = 1$, such a model is always coherent with respect to any set $\{p_1, \ldots, p_k\}$ that satisfies the following conditions:

(1) $\bigoplus_{i \neq j} p_i = \neg p_j$

(2) for any i, there exists $x \in [0,1]$ such that $p_i(x) = 1$.

This is a consequence of [13, Theorem 7] and Lemma 4.9, since in this case the assignment $p_i \mapsto \eta_i(x)$ gives a *partial state* for any $x \in [0,1]$, which can be extended to a σ-state on the whole algebra. The functions that satisfy conditions (1) and (2) above are called a *normal partition of the unit*.

5. Moment problem and exchangeability

We take inspiration from [7, Chapter VII] to give a version of the Hausdorff moment problem in the setting for σ-semisimple σ-complete Riesz MV-algebras. We will then use it to give a characterization of exchangeability for special sequences of observables. We refer to [16] for the missing notions on sequences of exchangeable observables.

In [17] the moment problem was proved for MV-algebras of continuous functions closed under product. We now tackle the case of measurable functions using Riesz tribes. The proof strategy of Theorem 5.1 is based on the one given in [17], the proof given by Feller in [7] and its reformulation in [18].

Let $\{m_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers, the Moment Problem on $I \subseteq \mathbb{R}$ consists on finding out the conditions on $\{m_k\}_{k\in\mathbb{N}}$ for which there exists a probability measure μ on I such that m_k is the k^{th} moment of μ, that is, $m_k = \int_I x^k d\mu$.

If $\{m_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers in $[0,1]$, we define the following, for any $r>0$, $k\geq 0$

$$\Delta^0 m_k = m_k, \quad \Delta^r m_k = \Delta^{r-1} m_{k+1} - \Delta^{r-1} m_k.$$

The sequence $\{m_k\}_{k\in\mathbb{N}}$ satisfies the *Hausdorff moment condition* if

$$m_0 = 1 \text{ and } (-1)^r \Delta^r m_k \geq 0 \text{ for any } r,k\geq 0. \tag{HMC}$$

In the following, for any $k\geq 1$ we denote by p_k the base polynomial x^k. Thus, p_k can be also thought as a function in $[0,1]^{[0,1]}$. Furthermore, we set $p_0(x)=1$ for any $x\in[0,1]$. We also remark that by Proposition 3.11, if V is a G_δ subset of $[0,1]$, any state on $IRL(1)|_V$ is a σ-state.

Theorem 5.1. *Let V be a G_δ subset of $[0,1]$ and take $A\simeq IRL(1)|_V$. There exists a σ-state $s: A\to[0,1]$ such that $s(p_k)=m_k$ if, and only, if the sequence $\{m_k\}_{k\in\mathbb{N}}$ satisfies the* Hausdorff moment condition*.*

Proof. Let s be a σ-state such that $s(p_k)=m_k$. By Theorem 2.2,

$$m_k = s(p_k) = \int_V p_k d\mu_s,$$

for any $k\in\mathbb{N}$, where μ_s is a probability measure on $(V,\mathcal{BA}(V))$. Then $m_0 = s(\mathbf{1}) = 1$.

Following the computations in [7, Chapter VII, Equation (1.7)] we can write

$$(-1)^r\Delta^r m_k = \sum_{h=0}^{r}\binom{r}{h}(-1)^h m_{k+h}.$$

Thus,

$$\begin{aligned}(-1)^r\Delta^r m_k &= \sum_{h=0}^{r}(-1)^h\binom{r}{h}\int_V p_{k+h}d\mu_s = \int_V\left[x^k\sum_{h=0}^{r}\binom{r}{h}(-1)^h x^h\right]d\mu_s\\ &= \int_V p_k(1-p_1)^r d\mu_s = s(p_k(1-p_1)^r)\geq 0,\end{aligned}$$

therefore the Hausdorff moment condition is satisfied.

To prove the converse direction, let $P([0,1])$ be the set of all polynomials $p:[0,1]\to\mathbb{R}$. Thus, $p=\sum_{i=1}^n a_i p_i$ for suitable scalars $a_1,\ldots,a_n\in\mathbb{R}$. Note that $P(V)=\{p|_V \mid p\in P([0,1])\}$ is a linear subspace of the Riesz space Baire(V) of restrictions to V of all bounded Baire-measurable functions in $\mathbb{R}^{[0,1]}$.

Let $t: P([0,1])\to\mathbb{R}$ be the linear functional defined by $t\left(\sum_{i=1}^n a_i p_i\right) = \sum_{i=1}^n a_i m_i$. Thus, since $m_0=1$, $t(\mathbf{c})=c$ for any constant $c\in\mathbb{R}$.

We now prove that t is a positive functional. This can be done exactly as in [18, Theorem 1]. We sketch the idea (which is in itself already contained in [7]) for its applicability to more general settings. We shall use *Bernstein polynomial*, see [7, Chapter VII.2], that are universal approximators for continuous functions. In particular, for any $p \in P([0,1])$, its Bernstein polynomial of degree n is

$$\begin{aligned} B_{p,n}(x) &= \sum_{r=1}^{n} \binom{n}{r} p\left(\frac{r}{n}\right) x^r (1-x)^{n-r} \\ &= \sum_{r=1}^{n} \binom{n}{r} p\left(\frac{r}{n}\right) \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j p_{r+j}. \end{aligned}$$

Thus, applying t to $B_{p,n}$ we get

$$t\left(B_{p,n}\right) = \sum_{r=1}^{n} \binom{n}{r} p\left(\frac{r}{n}\right) \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j m_{r+j}.$$

As seen before, $\sum_{j=0}^{n-r} \binom{n-r}{j}(-1)^j m_{r+j} = (-1)^{n-r} \Delta^{n-r} m_r$, which is greater of equal to 0 by hypothesis. Consequently, $t\left(B_{p,n}\right) \geq 0$ for any $n \in \mathbb{N}$ and any positive $p \in P([0,1])$. By [7, Chapter VII.2, Theorem 1], $B_{n,p}$ converges uniformely (and therefore, also pointwisely) to p and by [18, Lemma 2], $t(p) = \lim_{n \to +\infty} t(B_{n,p}) \geq 0$.

Now, since $V \subseteq [0,1]$, any $p \in P(V)$ is bounded and $t(p) = t(p - \inf_{x \in V}(p(x))) + \inf_{x \in V}(p(x))$. Since $p - \inf_{x \in V}(p(x)) \geq 0$, $t(p) \geq \inf_{x \in V}(p(x))$. Furthermore,

$$t(p) = -t(-p) \leq -\inf_{x \in V}(-p(x)) = \sup_{x \in V} p(x)$$

We have proved that t is a linear functional bounded by the supremum sublinear functional on $P(V)$. Hence, by the Hahn-Banach theorem, there exists $\bar{t}$: $\mathrm{Baire}(V) \to \mathbb{R}$ that extends t and it is bounded by the supremum on the whole $\mathrm{Baire}(V)$. Furthermore, $\bar{t}$ is still positive and $\bar{t}(\mathbf{1}) = t(\mathbf{1}) = 1$.

Denoted by $s : IRL(1)|_V \to [0,1]$ the restriction of $\bar{t}$ to A, we obtain the desired σ-state. □

We now see how to apply the moment problem to sequences of observables.

Assume again that V is a G_δ subset of $[0,1]$ and let $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ be a sequence of one-dimensional observables posed in the tribe $IRL(1)|_V$. That is, for any $n \in \mathbb{N}$, $\mathcal{X}_n \colon IRL(1) \to (IRL(1)|_V, s)$.

Assume also that the process is induced by $\{0,1\}$-valued measurable functions, that is $\mathcal{X}_n(a) = a \circ f_n$ with f_n boolean element of $IRL(1)|_V$. We can also assume that $f_n = \chi_{E_n}$ with $E_n \in \mathcal{BA}(V)$. We call such process a *boolean process of observables.*

Before giving the main result of this section, let us provide an example of a boolean process. Consider $([0,1], \mathcal{BO}[0,1]), \mu)$, where μ is the Lebesgue measure, and let $f_n \colon [0,1] \to \{0,1\}$ be the function that maps x in the n-th digit of its binary expansion. It is known, see for example [12, Lemma 3.20], that each f_n is a Bernoulli random variable with success rate $1/2$ and that the process $\{f_n\}_{n \in \mathbb{N}}$ is exchangeable. Thus, we might think of $\mathcal{X}_n(a) = a \circ f_n$ as the probability of success of an the experiment in which there are fuzzy aspects that need to be taken care of. For example, a could be a step function modeling the efficiency of a tool needed to perform the experiment.

Proposition 5.2. *With the above notations, the boolean process of observables* $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ *is weakly exchangeable if, and only if, for any* $n \in \mathbb{N}$ *and any* $k = 1, \dots, n$*, there exists a state* $t : IRL(1)|_V \to [0,1]$ *such that, for any set of indexes,* $i_1, \dots, i_k, i_{k+1}, \dots, i_n$

$$s(\chi_{C_k^n}) = t(p_n(1-p_1)^{n-k}), \tag{2}$$

where $C_k^n = E_{i_1} \cap \dots \cap E_{i_k} \cap (V \setminus E_{i_{k+1}}) \cap \dots \cap (V \setminus E_{i_n})$.

Proof. Assume that Equation (2) holds. By definition, see Section 2.3, the process is weakly exchangeable if for each $n \in \mathbb{N}$, and indexes $i_1, \dots, i_n$ and $j_1, \dots, j_n$ the joint distributions of the finite subsets $f_{i_1}, \dots, f_{i_n}$ and $f_{j_1}, \dots, f_{j_n}$ coincide. Note that such a joint distribution is given by

$$\mu_s\left(f_{i_1} = a_1, \dots, f_{i_n} = a_n\right) = \mu_s\left(\{x \in V \mid f_{i_1}(x) = a_1, \dots, f_{i_n}(x) = a_n\}\right),$$

with $a_1, \dots, a_n \in \{0,1\}$. Thus, we need to prove that $\mu_s\left(f_{i_1} = a_1, \dots, f_{i_n} = a_n\right) = \mu_s\left(f_{j_1} = a_1, \dots, f_{j_n} = a_n\right)$.

Assume that among the a_i's there are k ones and $n-k$ zeros and let $F_k^n = F_{i_1} \cap \dots \cap F_{i_n}$, where $F_{i_h} = E_{i_h}$ if $a_h = 1$ and $F_{i_h} = V \setminus E_{i_h}$ if $a_h = 0$. Similarly, let $G_k^n = G_{j_1} \cap \dots \cap G_{j_n}$, where $G_{j_h} = E_{j_h}$ if $a_h = 1$ and $G_{j_h} = V \setminus E_{j_h}$ if $a_h = 0$.

Then, by hypothesis, since each $f_n = \chi_{E_n}$,

$$\mu_s\left(f_{i_1} = a_1, \dots, f_{i_n} = a_n\right) = \mu_s(F_k^n) = s(\chi_{F_k^n}) = t(p_n(1-p_1)^{n-k}).$$

Analogously,

$$\mu_s\left(f_{j_1} = a_1, \dots, f_{j_n} = a_n\right) = s(\chi_{G_k^n}) = t(p_n(1-p_1)^{n-k}),$$

and the claim is settled.

Conversely, by exchangeability of the sequence $\{f_n\}_{n\in\mathbb{N}}$, the measure of any subset of type C_k^n only depends on n and k. Therefore, let $\mu_{k,n}$ be $\mu_s(C_k^n)$, where $0 \le k \le n$.

We will denote $c_0 = 1$ and $c_n = \mu_{n,n}$ for any $n \in \mathbb{N}$. By the probability laws of marginations and the hypothesis of exchangeability, we have

$$\begin{aligned}
\mu_{n-1,n} &= \mu_{n-1,n-1} - \mu_{n,n} = -\Delta c_{n-1},\\
\mu_{n-2,n} &= \mu_{n-2,n-1} - \mu_{n-1,n} = \Delta^2 c_{n-2},\\
\text{and further } \mu_{k,n} &= \mu_{k,n-1} - \mu_{k+1,n} = (-1)^{n-k}\Delta^{n-k} c_k.
\end{aligned}$$

Since $\mu_{k,n}$ are the values of a probability measure, they are non-negative and $(-1)^{n-k}\Delta^{n-k}c_k \ge 0$ for any $0 \le k \le n$. We can therefore apply Theorem 5.1. Hence, there exists a σ-state on $IRL(1)|_V$ such that $c_k = t(p_k)$.

We remark that $(-1)^r\Delta^r c_k = \sum_{h=0}^{r}\binom{r}{h}(-1)^h c_{k+h}$, where $r = n-k$. Then

$$\begin{aligned} s(\chi_{C_k^n}) =& \mu_s(C_k^n) = (-1)^r\Delta^r c_k = \sum_{h=0}^{r}(-1)^h\binom{r}{h}\int_V x^{k+h} d\mu_t \\ =& \int_V \left[x^k \sum_{h=0}^{r}\binom{r}{h}(-1)^h x^h\right] d\mu_t = \int_V x^k(1-x)^r d\mu_t \\ =& t(p_n(1-p_1)^{n-k}). \end{aligned}$$

We have proved that $s(\chi_{C_k^n}) = t(p_n(1-p_1)^{n-k})$ and the existence of t only depends on n and k. Therefore, the claim is settled. □

As mentioned in the introduction, this work is part of a larger project, whose broad goal is to give a logic-based approach to statistics. To achieve this goal, and to provide some effective reasoning mechanisms, more work is needed. In particular, future work will focus on expanding the discussion started in Section 4.1, where is given a first approach to coherence of an algebraic statistical model using de Finetti's characterization.

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