Summary of the PhD Thesis "MV-algebras with products: connecting the Pierce-Birkhoff conjecture with Łukasiewicz logic"
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The framework. In 1956, G. Birkhoff G. and R.S. Pierce [**1**] conjectured the following: *do the algebra of piecewise polynomial functions with real coefficients and the algebra of functions that can be written as Inf and Sup of finitely many polynomials coincide?*

The conjecture – that can be equivalently stated asking for the free n-generated lattice-ordered algebra to be isomorphic with the algebra of piecewise polynomial functions in n-variables – is still open. Many authors have approached the problem and the only answer so far is provided by L. Mahé in [**11**], who proved the conjecture for functions with at most two variables.

The aim of the thesis was to consider the problem from a fresh perspective. A seminal link with logic was provided by F. Montagna and G. Panti, that in [**14**] remarked that the study of free objects in varieties of MV-algebras with product is related to Pierce-Birkhoff conjecture. To give a more precise idea of how the framework of MV-algebras can be an appropriate choice for trying to solve the conjecture, some background is needed.

MV-algebras are the algebraic counterpart of Łukasiewicz logic and they are structures $(A, \oplus, ^*, 0)$ of type $(2, 1, 0)$ satisfying some appropriate axioms. The variety of MV-algebras is generated by $([0, 1], \oplus, ^*, 0)$ where $x \oplus y = \min(1, x + y)$ and $x^* = 1 - x$ for any $x, y \in [0, 1]$.

One of the main achievement in the theory of MV-algebras is the categorical equivalence with Abelian lattice-ordered groups with strong unit established by D. Mundici. A strong unit is a positive element u such that for any x in the group there exists an $n \in \mathbb{N}$ such that $x \leq nu$. As the unit interval $[0, 1]$ is closed under the product of real numbers, a fruitful research direction proved to be the idea of endowing an MV-algebra with products, either internal or scalar.

If the real product is interpreted as a binary operation on $[0, 1]$, the corresponding structures are MV-algebras A endowed with an operation $\cdot : A \times A \to A$. These algebras were introduced and studied by A. Di Nola and A. Dvurečenskij under the name of *PMV-algebras* in [**3**] and they are categorically equivalent to a class of lattice-ordered rings with strong unit. Particular important subclasses were further investigated by F. Montagna [**13, 15**], with the aim of studying the quasi-variety generated by $[0, 1]_{PMV} = ([0, 1], \oplus, \cdot, ^*, 0)$.

Another approach is to consider the real product on $[0, 1]$ as a multiplication with scalars in $[0, 1]$. The standard model in this case is $[0, 1]_{RMV} = ([0, 1], \oplus, ^*, \{\alpha \mid$

$\alpha \in [0,1]\}, 0)$ where $x \mapsto \alpha x$ is a unary operation, for any $\alpha \in [0,1]$. These structures are investigated by A. Di Nola and I. Leuştean under the name of *Riesz MV-algebras* [**4**] and they are categorically equivalent to Riesz spaces (vector lattices) with strong unit. The variety of Riesz MV-algebras is generated by the standard model $[0,1]_{RMV}$.

One of the main theorems of Łukasiewicz logic states that the functions corresponding to formulas of Łukasiewicz logic with n variables are exactly the continuous $[0,1]$-valued piecewise linear functions with integer coefficients defined on $[0,1]^n$. This is the so-called McNaughton theorem, and it can be seen as a *normal form theorem* for Łukasiewicz logic. A similar result was proved for the logical system that has Riesz MV-algebras as models; in this case the piecewise linear functions have real coefficients.

Clearly, the Pierce-Birkhoff conjecture has the same flavor of McNaughton theorem, but a more general setting was needed to formalize it logically: the algebra of piecewise polynomial functions with real coefficients is closed under both internal product and scalar product and none of the existing expansions of MV-algebras had this feature.

The original work. The work was developed following these main objectives:

(i) define an appropriate class of MV-algebras and rewrite the original Pierce-Birkhoff conjecture,

(ii) develop the technical tools needed to give a fresh look on the conjecture.

On the first goal, the class of fMV-algebras was defined and investigated. On the second goal, a thorough investigation of the tensor product of MV-algebra was carried. This allowed to place fMV-algebras in the general hierarchy of expansions of MV-algebras – via categorical adjunctions – and to rephrase the conjecture in a completely different way.

The class of fMV-algebras was the the one obtained by endowing MV-algebras with both the internal binary product and the scalar product (as a family of unary operations). The basic example of such an algebra is $[0,1]_{fMV} = ([0,1], \oplus, \cdot, ^*, \{\alpha \mid \alpha \in [0,1]\}, 0)$.

The first main results in the thesis are the proof of a categorical equivalence for the variety of all fMV-algebras and a characterization of the quasi-variety generated by $[0,1]_{fMV}$. In Theorem 2.2.1 fMV-algebras were proved to be categorical equivalent to f-algebras with strong unit, where an f-algebra is a lattice-ordered algebra that is a subdirect product of chains, while the quasi-variety generated by $[0,1]_{fMV}$ – denoted by $\mathbb{FR}^+$ in analogy with Montagna's PMV$^+$-algebras [**15**] –

was proved to be the class of fMV-algebras without nilpotent elements in Theorem 2.3.1.

The link between the Pierce-Birkhoff conjecture and the framework of fMV-algebras was provided by the algebra FR_n, the free algebra over n-generators in $\mathbb{FR}^+$. Indeed, Theorem 3.2.2 shows that FR_n is the algebra $PWL_u(n, \mathbb{R})$ of $[0,1]$-valued piecewise polynomial functions with real coefficients defined on $[0,1]^n$, for $n \leq 2$, and that $FR_n \subseteq PWL_u(n, \mathbb{R})$ in general, with the converse inclusion left as a conjecture: this gives a MV-algebraic rephrasing of the original conjecture. Note that such a rephrasing does not immediately imply nor it is implied by the Pierce-Birkhoff conjecture, as remarked after Conjecture 3.2.1: additional results are needed, and they are related to the possibility of extending a piecewise polynomial function from $[0,1]^n$ to $\mathbb{R}^n$.

As for MV-algebras and McNaughton functions, Conjecture 3.2.1 was related to a suitable logical system, denoted $\mathcal{FMVL}^+$, that has algebras in $\mathbb{FR}^+$ as models. $\mathcal{FMVL}^+$ is a conservative extension of Łukasiewicz logic and it is complete with respect to the unit interval $[0,1]_{fMV}$. Moreover, Conjecture 3.2.1 is *the normal form theorem* for $\mathcal{FMVL}^+$. The results mentioned so far are contained in the paper [**6**], and a survey of the results has appeared in [**9**].

The subsequent task was to tackle the conjecture: the core idea was to exploit the already known results for MV-algebras and Riesz MV-algebras, so that it would be possible to write the free n-generated fMV-algebra in terms of free MV-algebras and free Riesz MV-algebras. To do so, the machinery of choice was the one of the tensor product of MV-algebras.

In the context of lattice-ordered structures, several authors have worked on tensor product for ℓ-groups, Archimedean ℓ-groups and Riesz Spaces [**2, 5, 12**]. In the framework of MV-algebras, the definition of a tensor product has been instroduced by D. Mundici [**17**] in both standard and semisimple case (we recall that semisimple MV-algebras are equivalent to Archimedean ℓ-groups with strong unit).

The key technical result needed for the intended approach was the property of extensions of scalars (SEP). Such a property is trivial in the non-ordered case, while a complete proof for the case of ℓ-groups or Riesz Space is missing in literature. In Martinez's paper, the author consider the case of Riesz Spaces, but details are missing; in Steinberg's monography on lattice-ordered groups and modules the property is left as an exercise. After many attempts towards a proof, it was clear that most problems arise since the sum of two homomorphisms of ℓ-groups is not always an homomorphism of ℓ-groups. Nonetheless, the problem was solved for semisimple algebras and it was enough for such a logic-based approach, as free algebras are semisimple.

Thus, SEPs were proved for semisimple MV-algebras, Riesz MV-algebras and PMV-algebras (see Theorems 4.2.2 and 4.2.1, Corollary 4.2.1 and Proposition 4.2.1). Moreover, the same ideas led to the definition of the tensor PMV-algebra of a MV-algebra (see Definition 4.3.1 and Proposition 4.3.2), following the similar construction one can find in literature for the definition of the tensor algebra of a module.

Altogether these results led to two sets of adjunctions from semisimple MV-algebra to unital and semisimple fMV-algebra, one through PMV-algebras, one through Riesz MV-algebras. As a consequence, the free algebra FR_n was proved to be isomorphic with the tensor PMV-algebra of the free n-generated Riesz MV-algebra. Other applications of the results on tensor products led to the amalgamation property, proved for unital and semisimple PMV-algebras, unital and semisimple fMV-algebras and semisimple Riesz MV-algebras.

Finally, all results were trasfered (via categorical equivalence) to subclasses of ℓ-groups with strong unit, ℓ-rings with strong unit, f-algebras with strong unit, Riesz spaces with strong unit. The results on tensor product can be found in [**8, 10**].

Additional results: *f*MV-algebras and states. In the Handbook of Measure Theory, D. Mundici and B. Riečan left a list of open problems. One of them was the definition of an appropriate notion of stochastic independence for probability MV-algebras. As measure spaces are lattice-ordered algebras, fMV-algebras proved to be a good fitted framework to define stochastic independence, in which states ("additive" operators, see Definition 1.6.1) play the rôle of probability measures [**16**].

Theorem 5.1.1 provides an embedding theorem for any MV-algebra with a faithful state in a unit interval of suitable L^1-measure spaces. This theorem (that was build upon the state completion of an MV-algebras and a duality between state complete Riesz MV-algebras and L-measure spaces) led to Definition 5.2.1 and Theorem 5.2.1, in which a notion of stochastic independence is given and an universal property is proved, respectively.

The abovementioned embedding theorem entailed other two classical results: the well known Hölder's inequality and the Hausdorff moment problem, the latter for PMV-algebras and fMV-algebras. These results have been published in [**7**].

Conclusions and developments. The results contained in the thesis may help in casting a new light on the Pierce-Birkhoff conjecture: all attempts in solving the conjecture follow the path of topology and algebraic geometry, and this work can change the point of view on the problem. Indeed, following our chain of adjunctions, one gets that the free fMV-algebra is obtained applying the construction of the tensor PMV-algebra of a MV-algebra to the free Riesz MV-algebra. Since we already know that the free Riesz MV-algebra is the algebra of piecewise linear functions with real coefficients, our local version of the Pierce-Birkhoff conjecture, i.e. Conjecture 3.2.1, can be rewritten as *is any piecewise polynomial function with real coefficients a sum of products of piecewise linear functions with real coefficients?*, which can be interpreted as a "piecewise version" of the fundamental theorem of algebra.

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